From finite elements to Hybrid High-Order methods

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Setting

- Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be an open connected polytopal domain
- We focus on the Poisson problem: Find $u \in H_0^1(\Omega)$ s.t.

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \quad \forall v \in H_0^1(\Omega)$$

- The well-posedness of this problem hinges on the Poincaré inequality

$$\|v\|_{L^2(\Omega)} \lesssim \|\nabla v\|_{L^2(\Omega)^d} \quad \forall v \in H_0^1(\Omega)$$

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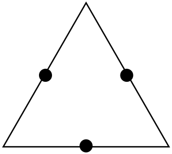
The Crouzeix–Raviart (CR) element



- Let $\mathcal{P}^1(T)$ be the space of affine functions on T
- Define the degrees of freedom $\sigma := (\sigma_F)_{F \in \mathcal{F}_T}$ s.t.

$$\sigma_F : \mathcal{P}^1(T) \ni v \mapsto v_F := \frac{1}{|F|} \int_F v \in \mathbb{R}$$

- The triplet $(T, \mathcal{P}^1(T), \sigma)$ is a FE in the sense of [Ciarlet, 2002]

A non-conforming FE scheme

- Let $\mathcal{T}_h$ be a conforming simplicial mesh
- Let $V_{h,0}$ be the global CR space on $\mathcal{T}_h$ with zero boundary DOFs
- We consider the scheme: Find $u_h \in V_{h,0}$ s.t.

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h = \int_\Omega f v_h \quad \forall v_h \in V_{h,0}$$

- Well-posedness follows from the discrete Poincaré inequality:

$$\|v_h\|_{L^2(\Omega)} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)^d} \qquad \forall v_h \in V_{h,0}$$

- Assuming for the exact solution $u \in H_0^1(\Omega) \cap H^2(\mathcal{T}_h)$, one can prove that

$$\|\nabla_h(u - u_h)\|_{L^2(\Omega)^2} \lesssim h|u|_{H^2(\mathcal{T}_h)}$$

Limitations

- Construction valid only on standard meshes
- Devising higher-order versions is not trivial
- **Can we remove these limitations?**

General meshes

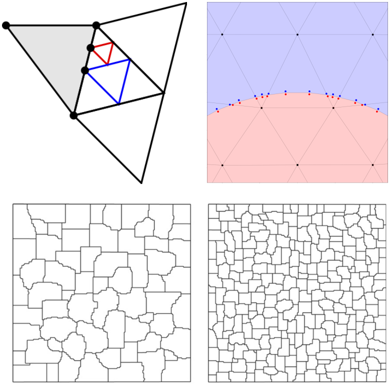


Figure: Examples of applications of general meshes, for which **a finite family of reference elements cannot be identified**

Shifting point of view I

- Let $v \in \mathcal{P}^1(T)$ and set, for all $F \in \mathcal{F}_T$,

$$\color{red}{v_F} := \frac{1}{|F|}\int_F v$$

- ∇v is fully determined in terms of the values $(v_F)_{F\in\mathcal{F}_T}$ by the equation

$$\int_T \nabla v \cdot \nabla w \overset{\text{IBP}}{=} \sum_{F\in\mathcal{F}_T}\int_F v\,\underbrace{(\nabla w \cdot n_{TF})}_{\in\mathcal{P}^0(F)} = \sum_{F\in\mathcal{F}_T}\int_F \color{red}{v_F}\color{black}(\nabla w \cdot n_{TF}) \quad \forall w \in \mathcal{P}^1(T)$$

- To express the average value of v in terms of $(v_F)_{F\in\mathcal{F}_T}$, we can write

$$\int_T v = \frac{1}{d}\int_T v\,\mathrm{div}(x - \overline{x}_T) \overset{\text{IBP}}{=} \frac{1}{d}\sum_{F\in\mathcal{F}_T}\int_F v\,\underbrace{(x-\overline{x}_T)\cdot n_{TF}}_{=:d_{TF}\in\mathbb{R}} = \sum_{F\in\mathcal{F}_T}\frac{d_{TF}}{d}\int_F \color{red}{v_F}$$

Shifting point of view II

- In conclusion, v is the unique solution of

$$\int_T \nabla v \cdot \nabla w = \sum_{F \in \mathcal{F}_T} \int_F v_F (\nabla w \cdot n_{TF}) \quad \forall w \in \mathcal{P}^1(T),$$

$$\int_T v = \sum_{F \in \mathcal{F}_T} \frac{d_{TF}}{d} \int_F v_F$$

- **This remains true if T is a (reasonable) polytope with planar faces!**

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A lowest-order hybrid space

- Denote by $\mathcal{M}_h = (\mathcal{T}_h, \mathcal{F}_h)$ a polytopal mesh of Ω
- We define the following space, spanned by vectors of local polynomials:

$$\underline{V}_h^0 := \left\{ \underline{v}_h = (v_F)_{F \in \mathcal{F}_h} \;:\; v_F \in \mathcal{P}^0(F) \text{ for all } F \in \mathcal{F}_h \right\}$$

- Smooth functions are interpolated through $\underline{I}_h^0 : H^1(\Omega) \to \underline{V}_h^0$ s.t.

$$\underline{I}_h^0 v := (\pi_F^0 v)_{F \in \mathcal{F}_h} \quad \forall v \in H^1(\Omega)$$

with $\pi_F^0 v := \frac{1}{|F|} \int_F v$ L^2-orthogonal projection of $v_{|F}$ on $\mathcal{P}^0(F)$

An affine potential reconstruction

- Let $T \in \mathcal{T}_h$ and denote by $\underline{V}_T^0$ the restriction of $\underline{V}_h^0$ to T
- Inspired by the previous remark, we let $p_T^1 : \underline{V}_T^0 \to \mathcal{P}^1(T)$ be s.t., for all $\underline{v}_T \in \underline{V}_T^0$,

$$\int_T \nabla p_T^1 \underline{v}_T \cdot \nabla w = \sum_{F \in \mathcal{F}_T} \int_F {\color{red} v_F} (\nabla w \cdot n_{TF}) \qquad \forall w \in \mathcal{P}^1(T),$$
$$\int_T p_T^1 \underline{v}_T = \sum_{F \in \mathcal{F}_T} \frac{d_{TF}}{d} \int_F {\color{red} v_F}$$

- With $\underline{I}_T^0$ restriction of $\underline{I}_h^0$ to T, it holds, by construction,

$$p_T^1(\underline{I}_T^0 v) = v \quad \forall v \in \mathcal{P}^1(T)$$

Extension to arbitrary order I

- We need to recover arbitrary-order polynomials in $\mathcal{P}^{k+1}(T)$, $k \geq 0$
- Given $Y \in \mathcal{T}_h \cup \mathcal{F}_h$ and $\ell \geq 0$, let $\pi_Y^\ell : L^2(Y) \to \mathcal{P}^\ell(Y)$ be s.t.

$$\int_Y \pi_Y^\ell v \, w = \int_Y v \, w \quad \forall w \in \mathcal{P}^\ell(Y)$$

- We notice that, for all $v \in \mathcal{P}^{k+1}(T)$,

$$\int_T \nabla v \cdot \nabla w = -\int_T \pi_T^{k-1} v \underbrace{\Delta w}_{\in \mathcal{P}^{k-1}(T)} + \sum_{F \in \mathcal{F}_T} \int_F \pi_F^k v \underbrace{(\nabla w \cdot n_{TF})}_{\in \mathcal{P}^k(F)} \quad \forall w \in \mathcal{P}^{k+1}(T)$$

- Moreover,

$$\int_T v = \int_T \pi_T^{k-1} v \text{ if } k \geq 1$$

Extension to arbitrary order II

- Hence, $v \in \mathcal{P}^{k+1}(T)$ is the unique solution of

$$\int_T \nabla v \cdot \nabla w = -\int_T \pi_T^{k-1} v \Delta w + \sum_{F \in \mathcal{F}_T} \int_F \pi_F^k v \, (\nabla w \cdot n_{TF}) \quad \forall w \in \mathcal{P}^{k+1}(T),$$

$$\int_T v = \begin{cases} \sum_{F \in \mathcal{F}_T} \frac{d_{TF}}{d} \int_F \pi_F^0 v & \text{if } k = 0, \\ \int_T \pi_T^{k-1} v & \text{if } k \geq 1 \end{cases}$$

- This suggests to consider the following extension of $\underline{V}_h^0$:

$$\underline{V}_h^k := \Big\{ \underline{v}_h = \big((v_T)_{T \in \mathcal{T}_h}, (v_F)_{F \in \mathcal{F}_h} \big) \; : v_T \in \mathcal{P}^{k-1}(T) \text{ for all } T \in \mathcal{T}_h, \\ v_F \in \mathcal{P}^k(F) \text{ for all } F \in \mathcal{F}_h \Big\}$$

- The natural interpolator $\underline{I}_h^k : H^1(\Omega) \to \underline{V}_h^k$ is s.t., for all $v \in H^1(\Omega)$,

$$\underline{I}_h^k v := ((\pi_T^{k-1} v)_{T \in \mathcal{T}_h}, (\pi_F^k v)_{F \in \mathcal{F}_h})$$

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An arbitrary-order potential reconstruction

- Denote by $\underline{V}_T^k$ the restriction of $\underline{V}_h^k$ to T
- We introduce $p_T^{k+1} : \underline{V}_T^k \to \mathcal{P}^{k+1}(T)$ s.t., for all $\underline{v}_T \in \underline{V}_T^k$,

$$\int_T \nabla p_T^{k+1} \underline{v}_T \cdot \nabla w = -\int_T v_T \Delta w + \sum_{F \in \mathcal{F}_T} \int_F v_F \, (\nabla w \cdot n_{TF}) \quad \forall w \in \mathcal{P}^{k+1}(T),$$

$$\int_T p_T^{k+1} \underline{v}_T = \begin{cases} \sum_{F \in \mathcal{F}_T} \frac{d_{TF}}{d} \int_F v_F & \text{if } k = 0, \\ \int_T v_T & \text{if } k \geq 1 \end{cases}$$

- **This problem has to be solved numerically inside each $T \in \mathcal{T}_h$!**
- By similar arguments as before, we have polynomial consistency:

$$p_T^{k+1}(\underline{I}_T^k v) = v \quad \forall v \in \mathcal{P}^{k+1}(T)$$

Elliptic projector

- The projector $\varpi_T^{k+1} := p_T^{k+1} \circ \underline{I}_T^k$ is characterized by: For all $v \in H^1(T)$,

$$\boxed{\begin{aligned} \int_T \nabla \varpi_T^{k+1} v \cdot \nabla w &= \int_T \nabla v \cdot \nabla w \quad \forall w \in \mathcal{P}^{k+1}(T), \\ \int_T \varpi_T^{k+1} v &= \begin{cases} \sum_{F \in \mathcal{F}_T} \frac{d_{TF}}{d} \int_F v & \text{if } k = 0, \\ \int_T v & \text{if } k \geq 1 \end{cases} \end{aligned}}$$

- This shows that it is an elliptic projector
- ϖ_T^{k+1} has optimal approximation properties, in particular:

$$\|\nabla(v - \varpi_T^{k+1} v)\|_{L^2(\partial T)^d} \lesssim h_T^{k+\frac{1}{2}} |v|_{H^{k+2}(T)} \quad \forall v \in H^{k+2}(T)$$

A discrete Poincaré inequality in hybrid spaces I

- For all $\underline{v}_h \in \underline{V}_h^k$, let $v_h \in L^2(\Omega)$ (not underlined) be s.t.

$$(v_h)_{|T} = v_T \quad \forall T \in \mathcal{T}_h$$

- Define the subspace of $\underline{V}_h^k$ with homogeneous boundary conditions

$$\underline{V}_{h,0}^k := \left\{ \underline{v}_h \in \underline{V}_h^k \ : \ v_F = 0 \text{ for all } F \in \mathcal{F}_h \text{ s.t. } F \subset \partial\Omega \right\}$$

- The equivalent of $|\cdot|_{H^1(\Omega)}$ on this space is $\|\cdot\|_{1,h}$ s.t., for all $\underline{v}_h \in \underline{V}_{h,0}^k$,

$$\|\underline{v}_h\|_{1,h}^2 := \sum_{T \in \mathcal{T}_h} \|\underline{v}_T\|_{1,T}^2,$$

$$\|\underline{v}_T\|_{1,T}^2 := \|\nabla v_T\|_{L^2(T)}^2 + \underbrace{h_T^{-1} \sum_{F \in \mathcal{F}_T} \|v_F - v_T\|_{L^2(F)}^2}_{=:|\underline{v}_T|_{1,\partial T}^2}$$

A discrete Poincaré inequality in hybrid spaces II

Lemma (Discrete Poincaré inequality in hybrid spaces)

For all $\underline{v}_h \in \underline{V}_{h,0}^k$, *it holds*

$$\|v_h\|_{L^2(\Omega)} \lesssim \|\underline{v}_h\|_{1,h}.$$

A discrete Poincaré inequality in hybrid spaces III

- By surjectivity of $\operatorname{div} : H^1(\Omega)^d \to L^2(\Omega)$, there is $\tau \in H^1(\Omega)^d$ s.t.

$$\operatorname{div} \tau = v_h \text{ and } \|\tau\|_{H^1(\Omega)^d} \lesssim \|v_h\|_{L^2(\Omega)}$$

- We thus have

$$\begin{aligned}
\|v_h\|^2_{L^2(\Omega)} &= \int_\Omega v_h \operatorname{div} \tau = \sum_{T \in \mathcal{T}_h} \int_T v_T \operatorname{div} \tau \\
&\overset{\text{IBP}}{=} \sum_{T \in \mathcal{T}_h} \left[-\int_T \nabla v_T \cdot \tau + \sum_{F \in \mathcal{F}_T} \int_F v_T (\tau \cdot n_{TF}) \right] \\
&= \sum_{T \in \mathcal{T}_h} \left[-\int_T \nabla v_T \cdot \tau + \sum_{F \in \mathcal{F}_T} \int_F (v_T - {\color{red} v_F})(\tau \cdot n_{TF}) \right]
\end{aligned}$$

A discrete Poincaré inequality in hybrid spaces IV

- Applying Cauchy–Schwarz to integrals and sums, we go on writing

$$\|v_h\|_{L^2(\Omega)}^2 \leq \left[\sum_{T\in\mathcal{T}_h}\left(\|\nabla v_T\|_{L^2(T)^d}^2 + h_T^{-1}\sum_{F\in\mathcal{F}_T}\|v_F - v_T\|_{L^2(F)}^2\right)\right]^{\frac{1}{2}}$$

$$\times\left[\sum_{T\in\mathcal{T}_h}\left(\|\tau\|_{L^2(T)^d}^2 + h_T\|\tau\|_{L^2(\partial T)}^2\right)\right]^{\frac{1}{2}}$$

- Recalling the definition of $\|\cdot\|_{1,h}$ and using trace inequalities, we get

$$\|v_h\|_{L^2(\Omega)}^2 \lesssim \|\underline{v}_h\|_{1,h}\,\|\tau\|_{H^1(\Omega)^d} \lesssim \|\underline{v}_h\|_{1,h}\,\|v_h\|_{L^2(\Omega)}$$

- Simplifying, the result follows

A coercive bilinear form

- We let $a_h : \underline{V}_h^k \times \underline{V}_h^k \to \mathbb{R}$ be s.t.

$$a_h(\underline{u}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T),$$

$$a_T(\underline{u}_T, \underline{v}_T) := \int_T \nabla p_T^{k+1} \underline{u}_T \cdot \nabla p_T^{k+1} \underline{v}_T + s_T(\underline{u}_T, \underline{v}_T)$$

- We assume that, for all $T \in \mathcal{T}_h$, s_T is s.t., for all $\underline{v}_T \in \underline{V}_T^k$,

$$\boxed{a_T(\underline{v}_T, \underline{v}_T) = \|\nabla p_T^{k+1} \underline{v}_T\|_{L^2(T)^d}^2 + s_T(\underline{v}_T, \underline{v}_T) \simeq \|\underline{v}_T\|_{1,T}^2} \qquad \text{(ST1)}$$

- Summing over $T \in \mathcal{T}_h$, we infer coercivity for a_h:

$$\|\underline{v}_h\|_{1,h}^2 \lesssim a_h(\underline{v}_h, \underline{v}_h) \quad \forall \underline{v}_h \in \underline{V}_{h,0}^k$$

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Discrete problem and basic error estimate I

- We consider the following scheme: Find $\underline{u}_h \in \underline{V}_{h,0}^k$ s.t.

$$a_h(\underline{u}_h, \underline{v}_h) = \int_\Omega f v_h \qquad \forall \underline{v}_h \in \underline{V}_{h,0}^k$$

- **Unlike FEM, we cannot plug u into a_h to prove convergence!**
- We instead study the discrete error defined as

$$\underline{e}_h \coloneqq \underline{u}_h - \underline{I}_h^k u$$

Discrete problem and basic error estimate II

- $\underline{e}_h$ is solution to the following equation:

$$a_h(\underline{e}_h, \underline{v}_h) = \int_\Omega f v_h - a_h(\underline{I}_h^k u, \underline{v}_h) =: \mathcal{E}_h(\underline{v}_h) \quad \forall \underline{v}_h \in \underline{V}_{h,0}^k,$$

  where $\mathcal{E}_h : \underline{V}_{h,0}^k \to \mathbb{R}$ is the consistency error linear form
- Denoting by $\|\cdot\|_{1,h,*}$ the norm dual to $\|\cdot\|_{1,h}$, we thus have

$$\|\underline{e}_h\|_{1,h}^2 \lesssim a_h(\underline{e}_h, \underline{e}_h) \le \|\mathcal{E}_h\|_{1,h,*} \|\underline{e}_h\|_{1,h},$$

  leading to the error estimate

$$\boxed{\|\underline{e}_h\|_{1,h} \lesssim \|\mathcal{E}_h\|_{1,h,*}}$$

Consistency error estimate I

- A complete characterization of s_T comes from the study of $\|\mathcal{E}_h\|_{1,h,*}$
- We assume in what follows that the exact solution satisfies

$$u \in H_0^1(\Omega) \cap H^{k+2}(\mathcal{T}_h)$$

- To estimate $\|\mathcal{E}_h\|_{1,h,*}$, we recast $\mathcal{E}_h$ to highlight the differences

$$(u - \varpi_T^{k+1} u)_{T \in \mathcal{T}_h},$$

which, by the approximation properties of ϖ_T^{k+1}, satisfy, for all $T \in \mathcal{T}_h$,

$$\|\nabla(u - \varpi_T^{k+1} u)\|_{L^2(\partial T)^d} \lesssim h_T^{k+\frac{1}{2}} |u|_{H^{k+2}(T)}$$

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Consistency error estimate II

- For the first contribution in $\mathcal{E}_h(\underline{v}_h)$, we write

$$\int_\Omega f v_h = -\int_\Omega \Delta u \, v_h \overset{\text{IBP}}{=} \sum_{T\in\mathcal{T}_h} \left(\int_T \nabla u \cdot \nabla v_T + \sum_{F\in\mathcal{F}_T} \int_F \nabla u \cdot n_{TF} ({\color{red}v_F} - v_T) \right)$$

- For the second, by definition of $p_T^{k+1}\underline{v}_T$ with $w = \varpi_T^{k+1} u$, we have

$$a_h(\underline{I}_h^k u, \underline{v}_h) = \sum_{T\in\mathcal{T}_h} \left(\int_T \nabla \varpi_T^{k+1} u \cdot \nabla v_T + \sum_{F\in\mathcal{F}_T} \int_F \nabla \varpi_T^{k+1} u \cdot n_{TF} (v_F - v_T) \right) + \sum_{T\in\mathcal{T}_h} s_T(\underline{I}_T^k u, \underline{v}_T)$$

Consistency error estimate III

- By the characterization of ϖ_T^{k+1}, since $v_T \in \mathcal{P}^{k-1}(T) \subset \mathcal{P}^{k+1}(T)$,

$$\int_T \nabla(u - \varpi_T^{k+1} u) \cdot \nabla v_T = 0$$

- Hence, gathering the previous results, we obtain

$$\mathcal{E}_h(u; \underline{v}_h) = \underbrace{\sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \int_F \nabla(u - \varpi_T^{k+1} u) \cdot n_{TF}(v_F - v_T)}_{\mathfrak{T}_1} \underbrace{- \sum_{T \in \mathcal{T}_h} s_T(\underline{I}_T^k u, \underline{v}_T)}_{\mathfrak{T}_2}$$

- For the first term, we have, using Cauchy–Schwarz inequalities,

$$\mathfrak{T}_1 \lesssim \left(\sum_{T \in \mathcal{T}_h} h_T \|\nabla(u - \varpi_T^{k+1} u)\|_{L^2(\partial T)}^2 \right)^{\frac{1}{2}} \|\underline{v}_h\|_{1,h} \lesssim h^{k+1} |u|_{H^{k+2}(\mathcal{T}_h)} \|\underline{v}_h\|_{1,h}$$

Consistency error estimate IV

- We would like the second term to scale in h^{k+1} as well
- This is the case if s_T is polynomially consistent:

$$\boxed{s_T(\underline{I}_T^k w, \underline{v}_T) = 0 \quad \forall (w, \underline{v}_T) \in \mathcal{P}^{k+1}(T) \times \underline{V}_T^k} \tag{ST2}$$

- **Can we find s_T satisfying (ST1) and (ST2)?**

Local stabilization I

Proposition (Structure of the stabilization bilinear form)

Let $\underline{\delta}_T^k : \underline{V}_T^k \to \underline{V}_T^k$ *be s.t., for all* $\underline{v}_T \in \underline{V}_T^k$,

$$\underline{\delta}_T^k \underline{v}_T \coloneqq \underline{I}_T^k p_T^{k+1} \underline{v}_T - \underline{v}_T.$$

(ST1)-(ST2) *hold iff there is a symmetric bilinear form* $\mathcal{S}_T : \underline{V}_T^k \times \underline{V}_T^k \to \mathbb{R}$ *s.t.*

$$s_T(\underline{v}_T, \underline{w}_T) = \mathcal{S}_T(\underline{\delta}_T^k \underline{v}_T, \underline{\delta}_T^k \underline{w}_T) \quad \forall (\underline{v}_T, \underline{w}_T) \in \underline{V}_T^k \times \underline{V}_T^k$$

and

$$|\underline{v}_T|_{1,\partial T}^2 \lesssim \mathcal{S}_T(\underline{v}_T, \underline{v}_T) \lesssim \|\underline{v}_T\|_{1,T}^2 \qquad \forall \underline{v}_T \in \underline{V}_T^k.$$

Local stabilization II

Example (Original HHO stabilization)

The original HHO stabilization of [DP, Ern, Lemaire, 2014] is obtained setting

$$\mathcal{S}_T(\underline{w}_T, \underline{v}_T) \coloneqq h_T^{-1} \sum_{F \in \mathcal{F}_T} \int_F (w_F - w_T)(v_F - v_T).$$

Example (VEM-type stabilization)

A stabilization inspired by Virtual Elements [Beirão da Veiga et al., 2013] is obtained with

$$\mathcal{S}_T(\underline{w}_T, \underline{v}_T) \coloneqq h_T^{-2} \int_T w_T\, v_T + h_T^{-1} \sum_{F \in \mathcal{F}_T} \int_F w_F\, v_F.$$

A variation based on a gradient reconstruction

- Alternatively, one can reconstruct the gradient instead of the potential
- Specifically, let $G_T^k : \underline{V}_T^k \to \mathcal{P}^k(T)^d$ be s.t., for all $\underline{v}_T \in \underline{V}_T^k$,

$$\int_T G_T^k \underline{v}_T \cdot \tau = -\int_T v_T \operatorname{div} \tau + \sum_{F \in \mathcal{F}_T} \int_F v_F (\tau \cdot n_{TF}) \quad \forall \tau \in \mathcal{P}^k(T)^d$$

- Given s_T satisfying (ST1)-(ST2), a different method is obtained setting

$$a_T(\underline{u}_T, \underline{v}_T) := \int_T G_T^k \underline{u}_T \cdot G_T^k \underline{v}_T + s_T(\underline{u}_T, \underline{v}_T)$$

- This variation is better suited to treat locally variable diffusion
- For further details, see, e.g., [Di Pietro and Droniou, 2020, Section 4.2]

Other spaces

- (Much) more complicated spaces/reconstructions exist
- **Need for a polytopal-oriented DS(E)L!**

Space	Vertices V	Edges E	Element T
$\underline{X}_{\mathrm{grad},F}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(T)$
$\underline{X}_{\mathrm{curl},F}$		$\mathcal{P}^{k}(E)$	$\mathcal{R}^{k-1}(T) \times \mathcal{R}^{\mathrm{c},k}(T)$
$\mathcal{P}^{k}(T)$			$\mathcal{P}^{k}(T)$

Table: Examples of spaces appearing in the two-dimensional Discrete de Rham method of [Di Pietro and Droniou, 2023]. Above, $\mathcal{R}^{k-1}(T) \coloneqq \mathrm{rot}\,\mathcal{P}^{k}(T)$ and $\mathcal{R}^{\mathrm{c},k}(T) \coloneqq (x - x_T)\mathcal{P}^{k-1}(T)$.

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Thank you for your attention!

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