Polytopal approximations of Hilbert complexes

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NEMESIS

New generation methods
for numerical simulations

Montpellier, 19/06/2024

IMAG
UNIVERSITÉ DE MONTPELLIER
Funded by the European Union
European Research Council

Research Cluster 1: Discrete Hilbert complexes

- Polytopal de Rham complexes
  - Construction of a Rosetta stone to bridge the virtual and fully discrete approaches
  - Development of novel polytopal de Rham complexes with competitive features
  - Application of these new complexes to model problems
  - Development of **Polytopal Exterior Calculus (PEC)**
- Extended polytopal complexes
  - Novel polytopal elasticity complexes, applied to solid- and fluid-mechanics
  - Novel polytopal Hessian complexes, applied to linear elasticity and general relativity
  - Generalization of PC to cover extended polytopal complexes

Outline

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Setting

- Let $\Omega \subset \mathbb{R}^3$ be a connected polyhedral domain with Betti numbers b_i
- $b_0 = 1$ (number of connected components) and $b_3 = 0$ (since $d = 3$)
- b_1 and b_2 respectively account for the number of tunnels and voids

$(b_0, b_1, b_2, b_3) = (1, 1, 0, 0)$

$(b_0, b_1, b_2, b_3) = (1, 0, 1, 0)$

- Assume for the moment that Ω has trivial topology, i.e.,

$$b_1 = b_2 = 0$$

Two model problems

- We consider two examples of PDEs set in $H^1(\Omega)$, $H(\mathrm{curl};\Omega)$, and $H(\mathrm{div};\Omega)$
- The Stokes problem: Find $(u,p) \in H(\mathrm{curl};\Omega) \times H^1(\Omega)$ s.t. $\int_\Omega p = 0$ and

$$\begin{aligned} \nu \int_\Omega \mathrm{curl}\, u \cdot \mathrm{curl}\, v + \int_\Omega \mathrm{grad}\, p \cdot v &= \int_\Omega f \cdot v \quad && \forall v \in H(\mathrm{curl};\Omega), \\ -\int_\Omega u \cdot \mathrm{grad}\, q &= 0 && \forall q \in H^1(\Omega) \end{aligned}$$

- The magnetostatics problem: Find $(H,A) \in H(\mathrm{curl};\Omega) \times H(\mathrm{div};\Omega)$ s.t.

$$\begin{aligned} \mu \int_\Omega H \cdot \tau - \int_\Omega A \cdot \mathrm{curl}\, \tau &= 0 && \forall \tau \in H(\mathrm{curl};\Omega), \\ \int_\Omega \mathrm{curl}\, H \cdot v + \int_\Omega \mathrm{div}\, A\, \mathrm{div}\, v &= \int_\Omega J \cdot v \quad && \forall v \in H(\mathrm{div};\Omega) \end{aligned}$$

A unified view

- The above problems are mixed formulations involving two fields:
  Find $(\sigma, u) \in \Sigma \times U$ s.t.

  $$\begin{aligned} a(\sigma,\tau) + b(\tau,u) &= f(\tau) \quad \forall \tau \in \Sigma, \\ -b(\sigma,v) + c(u,v) &= g(v) \quad \forall v \in U, \end{aligned}$$

  or, equivalently, in variational formulation,

  $$\mathcal{A}((\sigma,u),(\tau,v)) = f(\tau) + g(v) \qquad \forall (\tau,v) \in \Sigma \times U$$

  with

  $$\mathcal{A}((\sigma,u),(\tau,v)) := a(\sigma,\tau) + b(\tau,u) - b(\sigma,v) + c(u,v)$$

- Well-posedness holds under an inf-sup condition on $\mathcal{A}$

A unified tool for well-posedness: The de Rham complex

$$H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl};\Omega) \xrightarrow{\text{curl}} H(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow \{0\}$$

- Key properties, possibly depending on the topology of Ω:

$$\begin{aligned} \operatorname{Im}\text{grad} &\subset \operatorname{Ker}\text{curl}, \\ \operatorname{Im}\text{curl} &\subset \operatorname{Ker}\text{div}, \\ \Omega \subset \mathbb{R}^3\ (b_3 = 0) \implies \operatorname{Im}\text{div} &= L^2(\Omega) \quad \text{(magnetostatics)} \end{aligned}$$

A unified tool for well-posedness: The de Rham complex

$$H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl};\Omega) \xrightarrow{\text{curl}} H(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow \{0\}$$

- Key properties, possibly depending on the topology of Ω:

no tunnels crossing Ω ($b_1 = 0$) $\Longrightarrow$ $\text{Im grad} = \text{Ker curl}$ (Stokes)

no voids contained in Ω ($b_2 = 0$) $\Longrightarrow$ $\text{Im curl} = \text{Ker div}$ (magnetostatics)

$\Omega \subset \mathbb{R}^3$ ($b_3 = 0$) $\Longrightarrow$ $\text{Im div} = L^2(\Omega)$ (magnetostatics)

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- Key properties, possibly depending on the topology of Ω:

$$\text{no tunnels crossing } \Omega\ (b_1 = 0) \implies \text{Im grad} = \text{Ker curl} \quad \text{(Stokes)}$$

$$\text{no voids contained in } \Omega\ (b_2 = 0) \implies \text{Im curl} = \text{Ker div} \quad \text{(magnetostatics)}$$

$$\Omega \subset \mathbb{R}^3\ (b_3 = 0) \implies \text{Im div} = L^2(\Omega) \quad \text{(magnetostatics)}$$

- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\mathcal{H}_1 := \text{Ker curl}/\text{Im grad} \quad \text{and} \quad \mathcal{H}_2 := \text{Ker div}/\text{Im curl}$$

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$\Omega \subset \mathbb{R}^3$ ($b_3 = 0$) $\Longrightarrow$ $\text{Im div} = L^2(\Omega)$ (magnetostatics)

- When $b_1 \neq 0$ or $b_2 \neq 0$, de Rham's cohomology characterizes

$$\mathcal{H}_1 := \text{Ker curl}/\text{Im grad} \quad \text{and} \quad \mathcal{H}_2 := \text{Ker div}/\text{Im curl}$$

- **Emulating these algebraic properties is key for stable discretizations**

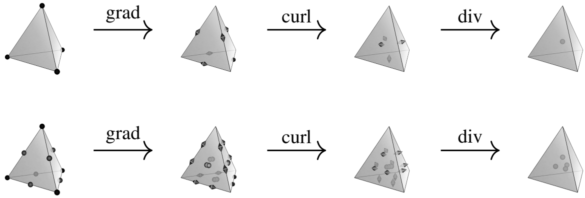
The Finite Element way

- Trimmed FE complexes on a tetrahedron T[1]: For any $k \geq 1$

$k = 1$

$k = 2$

$\dots$

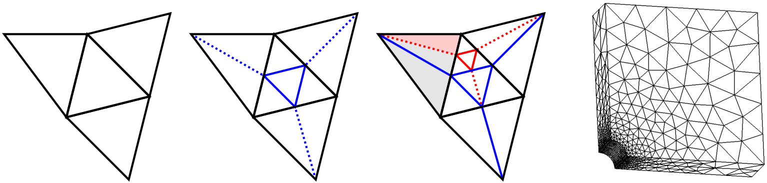


- On a conforming tetrahedral meshes $\mathcal{T}_h$, these spaces can be glued together

$$\begin{array}{ccccccc}
H^1(\Omega) & \xrightarrow{\text{grad}} & H(\text{curl};\Omega) & \xrightarrow{\text{curl}} & H(\text{div};\Omega) & \xrightarrow{\text{div}} & L^2(\Omega) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathcal{P}^k_{\text{c}}(\mathcal{T}_h) & \xrightarrow{\text{grad}} & \mathcal{N}^k(\mathcal{T}_h) & \xrightarrow{\text{curl}} & \mathcal{RT}^k(\mathcal{T}_h) & \xrightarrow{\text{div}} & \mathcal{P}^{k-1}(\mathcal{T}_h)
\end{array}$$

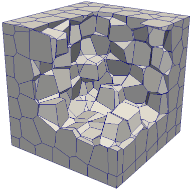
[1][Raviart and Thomas, 1977, Nédélec, 1980]

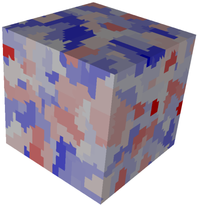
Limitations

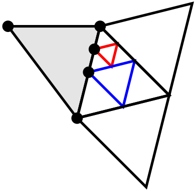


- Approach limited to conforming meshes with standard elements
  - $\Longrightarrow$ Local refinement requires to trade mesh size for mesh quality
  - $\Longrightarrow$ Complex geometries may require a large number of elements
  - $\Longrightarrow$ The element shape cannot be adapted to the solution
- The extension to advanced complexes is also not straightforward

Polytopal approaches







- Key idea: replace spaces and, possibly, operators by discrete counterparts
- Support of polyhedral meshes and high-order
- Higher-level point of view, possibly resulting in leaner constructions
- Several strategies to reduce the number of unknowns on general shapes
- Agglomeration-based[2] techniques (see P. Antonietti's presentation)

[2][Bassi et al., 2012], [Antonietti et al., 2013]

Outline

Isomorphism in cohomology

Theorem (Complexes with isomorphic cohomologies[3])

$$\begin{array}{ccccccc} \cdots & \longrightarrow & V_i & \xrightarrow{d_i} & V_{i+1} & \longrightarrow & \cdots \\ & & E_i \big\downarrow \big\uparrow R_i & & E_{i+1} \big\downarrow \big\uparrow R_{i+1} & & \\ \cdots & \longrightarrow & W_i & \xrightarrow{\partial_i} & W_{i+1} & \longrightarrow & \cdots \end{array}$$

Assume that reduction R and extension E are s.t., for all i,

- $R_i E_i = \mathrm{Id}_{W_i}$;
- $(E_{i+1} R_{i+1} - \mathrm{Id}_{V_{i+1}}) \operatorname{Ker} d_{i+1} \subset \operatorname{Im} d_i$;
- $\partial_i E_i = E_{i+1} d_i$ and $d_i R_i = R_{i+1} \partial_i$.

Then, the sequences are complexes with isomorphic cohomologies.

[3][DP, Droniou and Pitassi, 2023]

Cohomology of the trimmed FE complex

- Let $\mathcal{M}_h := \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h \cup \mathcal{V}_h$ be the simplicial complex underlying a FE mesh
- Denoting by ∂ the coboundary operator, we have

$$\begin{array}{ccccccc}
\mathcal{V}_h^* & \xrightarrow{\partial_0} & \mathcal{E}_h^* & \xrightarrow{\partial_1} & \mathcal{F}_h^* & \xrightarrow{\partial_2} & \mathcal{V}_h^* \\
\kappa_{0,h} \cong \uparrow & & \kappa_{1,h} \cong \uparrow & & \kappa_{2,h} \cong \uparrow & & \kappa_{3,h} \cong \uparrow \\
\mathcal{P}_{\mathrm{c}}^1(\mathcal{T}_h) & \xrightarrow{\mathrm{grad}} & \mathcal{N}^1(\mathcal{T}_h) & \xrightarrow{\mathrm{curl}} & \mathcal{RT}^1(\mathcal{T}_h) & \xrightarrow{\mathrm{div}} & \mathcal{P}^0(\mathcal{T}_h) \\
\downarrow \; \uparrow I^1_{\mathrm{grad},h} & & \downarrow \; \uparrow I^1_{\mathrm{curl},h} & & \downarrow \; \uparrow I^1_{\mathrm{div},h} & & \downarrow \; \uparrow \pi_h^0 \\
\mathcal{P}_{\mathrm{c}}^k(\mathcal{T}_h) & \xrightarrow{\mathrm{grad}} & \mathcal{N}^k(\mathcal{T}_h) & \xrightarrow{\mathrm{curl}} & \mathcal{RT}^k(\mathcal{T}_h) & \xrightarrow{\mathrm{div}} & \mathcal{P}^{k-1}(\mathcal{T}_h)
\end{array}$$

with κ_h de Rham map, $I^1_{\bullet,h}$ interpolator, and π_h^0 L^2-orthogonal projector

- By the de Rham Theorem, the first two rows have isomorphic cohomologies

Cohomology of the trimmed FE complex

- Let $\mathcal{M}_h := \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h \cup \mathcal{V}_h$ be the simplicial complex underlying a FE mesh
- Denoting by ∂ the coboundary operator, we have

$$
\begin{array}{ccccccc}
\mathcal{V}_h^* & \xrightarrow{\partial_0} & \mathcal{E}_h^* & \xrightarrow{\partial_1} & \mathcal{F}_h^* & \xrightarrow{\partial_2} & \mathcal{V}_h^* \\
\kappa_{0,h}\,\cong\uparrow & & \kappa_{1,h}\,\cong\uparrow & & \kappa_{2,h}\,\cong\uparrow & & \kappa_{3,h}\,\cong\uparrow \\
\mathcal{P}_{\rm c}^1(\mathcal{T}_h) & \xrightarrow{\rm grad} & \mathcal{N}^1(\mathcal{T}_h) & \xrightarrow{\rm curl} & \mathcal{RT}^1(\mathcal{T}_h) & \xrightarrow{\rm div} & \mathcal{P}^0(\mathcal{T}_h) \\
\updownarrow I^1_{{\rm grad},h} & & \updownarrow I^1_{{\rm curl},h} & & \updownarrow I^1_{{\rm div},h} & & \updownarrow \pi_h^0 \\
\mathcal{P}_{\rm c}^k(\mathcal{T}_h) & \xrightarrow{\rm grad} & \mathcal{N}^k(\mathcal{T}_h) & \xrightarrow{\rm curl} & \mathcal{RT}^k(\mathcal{T}_h) & \xrightarrow{\rm div} & \mathcal{P}^{k-1}(\mathcal{T}_h)
\end{array}
$$

with κ_h de Rham map, $I^1_{\bullet,h}$ interpolator, and π_h^0 L^2-orthogonal projector

- By the de Rham Theorem, the first two rows have isomorphic cohomologies
- **The two bottom rows fulfill the assumptions of the theorem!**

Shifting point of view I

- Denote by "dofs" the standard FE degrees of freedom
- By unisolvency, we have

$$
\begin{array}{ccccccc}
\mathcal{V}_h^* & \xrightarrow{\partial_0} & \mathcal{E}_h^* & \xrightarrow{\partial_1} & \mathcal{F}_h^* & \xrightarrow{\partial_2} & \mathcal{V}_h^* \\
\kappa_{0,h} \cong \Big\uparrow & & \kappa_{1,h} \cong \Big\uparrow & & \kappa_{2,h} \cong \Big\uparrow & & \kappa_{3,h} \cong \Big\uparrow \\
\mathcal{P}_{\mathrm{c}}^1(\mathcal{T}_h) & \xrightarrow{\text{grad}} & \mathcal{N}^1(\mathcal{T}_h) & \xrightarrow{\text{curl}} & \mathcal{RT}^1(\mathcal{T}_h) & \xrightarrow{\text{div}} & \mathcal{P}^0(\mathcal{T}_h) \\
\text{dofs}^{-1} \cong \Big\uparrow & & \text{dofs}^{-1} \cong \Big\uparrow & & \text{dofs}^{-1} \cong \Big\uparrow & & \text{dofs}^{-1} \cong \Big\uparrow \\
\mathbb{R}^{\mathcal{V}_h} & & \mathbb{R}^{\mathcal{E}_h} & & \mathbb{R}^{\mathcal{F}_h} & & \mathbb{R}^{\mathcal{T}_h}
\end{array}
$$

Shifting point of view II

- In the previous diagram, we can erase the middle row
- Set $K_h \coloneqq \text{dofs}^{-1} \circ \kappa_h$, i.e., for all $(\underline{q}_h, \underline{v}_h, \underline{w}_h, \underline{r}_h) \in \mathbb{R}^{\mathcal{V}_h} \times \mathbb{R}^{\mathcal{E}_h} \times \mathbb{R}^{\mathcal{F}_h} \times \mathbb{R}^{\mathcal{T}_h}$,

$$
\begin{aligned}
K_{0,h}\underline{q}_h(V) &\coloneqq q_V \quad \forall V \in \mathcal{V}_h, & K_{1,h}\underline{v}_h(E) &\coloneqq |E| v_E \quad \forall E \in \mathcal{E}_h, \\
K_{2,h}\underline{w}_h(F) &\coloneqq |F| v_F \quad \forall F \in \mathcal{F}_h, & K_{3,h} r_h &\coloneqq |T| r_T \quad \forall T \in \mathcal{T}_h
\end{aligned}
$$

- These maps induce the following isomorphisms:

$$
\begin{array}{ccccccc}
\mathcal{V}_h^* & \xrightarrow{\partial_0} & \mathcal{E}_h^* & \xrightarrow{\partial_1} & \mathcal{F}_h^* & \xrightarrow{\partial_2} & \mathcal{V}_h^* \\
{\scriptstyle K_{0,h}} \cong \uparrow & & {\scriptstyle K_{1,h}} \cong \uparrow & & {\scriptstyle K_{2,h}} \cong \uparrow & & {\scriptstyle K_{3,h}} \cong \uparrow \\
\mathbb{R}^{\mathcal{V}_h} & & \mathbb{R}^{\mathcal{E}_h} & & \mathbb{R}^{\mathcal{F}_h} & & \mathbb{R}^{\mathcal{T}_h}
\end{array}
$$

- **Can we complete the bottom row to form a complex?**

Shifting point of view III

- Define the following discrete gradient, curl, and divergence operators:

$$\underline{G}_h^0 \underline{q}_h := K_{1,h}^{-1} \partial_0 K_{0,h}, \quad \underline{C}_h^0 \underline{v}_h := K_{2,h}^{-1} \partial_1 K_{1,h}, \quad \underline{D}_h^0 \underline{w}_h := K_{3,h}^{-1} \partial_2 K_{2,h},$$

- Notice that, by construction,

$$\underline{C}_h^0 \circ \underline{G}_h^0 = \underline{0} \text{ and } \underline{D}_h^0 \circ \underline{C}_h^0 = \underline{0}$$

- Hence, we have two complexes with isomorphic cohomologies:

$$\begin{array}{ccccccc}
\mathcal{V}_h^* & \xrightarrow{\partial_0} & \mathcal{E}_h^* & \xrightarrow{\partial_1} & \mathcal{F}_h^* & \xrightarrow{\partial_2} & \mathcal{V}_h^* \\
\uparrow {\scriptstyle K_{0,h} \atop \cong} & & \uparrow {\scriptstyle K_{1,h} \atop \cong} & & \uparrow {\scriptstyle K_{2,h} \atop \cong} & & \uparrow {\scriptstyle K_{3,h} \atop \cong} \\
\mathbb{R}^{\mathcal{V}_h} & \xrightarrow{\underline{G}_h^0} & \mathbb{R}^{\mathcal{E}_h} & \xrightarrow{\underline{C}_h^0} & \mathbb{R}^{\mathcal{F}_h} & \xrightarrow{\underline{D}_h^0} & \mathbb{R}^{\mathcal{T}_h}
\end{array}$$

- **Still true with $\mathcal{M}_h$ CW complex associated to a polyhedral mesh!**

A closer look to the discrete operators

- $\underline{G}_h^0$, $\underline{C}_h^0$, and $\underline{D}_h^0$ are actually the mimetic operators[4]:

$$\underline{G}_h^0 \underline{q}_h := \left(G_E^0 \underline{q}_E = \frac{q_{V_2} - q_{V_1}}{|E|} \right)_{E \in \mathcal{E}_h},$$

$$\underline{C}_h^0 \underline{v}_h := \left(C_F^0 \underline{v}_F = -\frac{1}{|F|} \sum_{E \in \mathcal{E}_F} \omega_{FE} |E| v_E \right)_{F \in \mathcal{F}_h},$$

$$\underline{D}_h^0 \underline{w}_h := \left(D_T^0 \underline{w}_T = \frac{1}{|T|} \sum_{F \in \mathcal{F}_T} \omega_{TF} |F| w_F \right)_{T \in \mathcal{T}_h}$$

- These operators are polynomially exact

[4] See, e.g., [Beirão da Veiga et al., 2014] and [Bonelle and Ern, 2014]

The arbitrary-order case $k \geq 0$

$$\underline{X}^k_{\mathrm{grad},h} \xrightarrow{\underline{G}^k_h} \underline{X}^k_{\mathrm{curl},h} \xrightarrow{\underline{C}^k_h} \underline{X}^k_{\mathrm{div},h} \xrightarrow{D^k_h} \mathcal{P}^{k-1}(\mathcal{T}_h)$$

	V	E	F	T
$\underline{X}^k_{\mathrm{grad},h}$	$\mathbb{R}$	$\mathcal{P}^{k-1}(E)$	$\mathcal{P}^{k-1}(F)$	$\mathcal{P}^{k-1}(T)$
$\underline{X}^k_{\mathrm{curl},h}$	–	$\mathcal{P}^{k}(E)$	$\mathcal{RT}^{k}(F)$	$\mathcal{RT}^{k}(T)$
$\underline{X}^k_{\mathrm{div},h}$	–	–	$\mathcal{P}^{k}(F)$	$\mathcal{N}^{k}(T)$
$\underline{X}^k_{\mathrm{L},h}$	–	–	–	$\mathcal{P}^{k}(T)$

- Discrete de Rham (DDR) [DP et al., 2020, DP and Droniou, 2023a]
- Serendipity version [DP and Droniou, 2023b][5]
- Extension to other complexes and applications in various subsequent papers

[5] See [Beirão da Veiga et al., 2018b] for a preliminary work on this subject

An example: The arbitrary order curl space I

- The face curl $C_F^k : \underline{X}_{\mathrm{curl},F}^k \to \mathcal{P}^k(F)$ is s.t.

$$\int_F C_F^k \underline{v}_F\, q = \int_F v_F \cdot \mathrm{rot}_F\, q - \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E\, q \quad \forall q \in \mathcal{P}^k(F)$$

- The tangent trace $\gamma_F^k : \underline{X}_{\mathrm{curl},F}^k \to \mathcal{P}^k(F)^2$ is s.t., $\forall (r, w) \in \mathcal{P}^{0,k+1}(F) \times \mathcal{R}^{\mathrm{c},k}(F)$,

$$\int_F \gamma_F^k \underline{v}_F \cdot (\mathrm{rot}_F\, r + w) = \int_F C_F^k \underline{v}_F\, r + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E v_E\, r + \int_F v_F \cdot w$$

- From γ_F^k, we build the element curl $C_T^k : \underline{X}_{\mathrm{curl},h}^k \to \mathcal{P}^k(T)^3$ similarly to C_F^k and set

$$\begin{aligned} \underline{C}_h^k : \underline{X}_{\mathrm{curl},h}^k &\to \underline{X}_{\mathrm{div},h}^k \\ \underline{v}_h &\mapsto \left((\pi_{\mathcal{N}^k(T)} C_T^k \underline{v}_T)_{T \in \mathcal{T}_h}, (C_F^k \underline{v}_F)_{F \in \mathcal{F}_h} \right) \end{aligned}$$

An example: The arbitrary order curl space II

- From C_T^k and γ_F^k, we build an element potential $P^k_{\mathrm{curl},T} : \underline{X}^k_{\mathrm{curl},T} \to \mathcal{P}^k(T)^3$
- The local L^2-product in $\underline{X}^k_{\mathrm{curl},T}$ is

$$(\underline{w}_T, \underline{v}_T)_{\mathrm{curl},T} \coloneqq \int_T P^k_{\mathrm{curl},T}\underline{w}_T \cdot P^k_{\mathrm{curl},T}\underline{v}_T + \text{stab.}$$

  where stab. penalizes $\underline{v}_T - \underline{I}^k_{\mathrm{curl},T}P^k_{\mathrm{curl},T}\underline{v}_T$ in a least-square sense
- The global discrete L^2-product is obtained assemblying element-wise:

$$(\underline{w}_h, \underline{v}_h)_{\mathrm{curl},h} \coloneqq \sum_{T\in\mathcal{T}_h} (\underline{w}_T, \underline{v}_T)_{\mathrm{curl},T}$$

An example of numerical scheme

- Let us consider again the magnetostatics problem:
  Find $(H, A) \in H(\mathrm{curl};\Omega) \times H(\mathrm{div};\Omega)$ s.t.

$$
\begin{aligned}
\mu \int_\Omega H \cdot \tau - \int_\Omega A \cdot \mathrm{curl}\,\tau &= 0 && \forall \tau \in H(\mathrm{curl};\Omega),\\
\int_\Omega \mathrm{curl}\,H \cdot v + \int_\Omega \mathrm{div}\,A\,\mathrm{div}\,v &= \int_\Omega J \cdot v && \forall v \in H(\mathrm{div};\Omega)
\end{aligned}
$$

- A DDR scheme for this problem is obtained with obvious substitutions:
  Find $(\underline{H}_h, \underline{A}_h) \in \underline{X}^k_{\mathrm{curl},h} \times \underline{X}^k_{\mathrm{div},h}$ s.t.

$$
\begin{aligned}
\mu(\underline{H}_h, \underline{\tau}_h)_{\mathrm{curl},h} - (\underline{A}_h, \underline{C}^k_h \underline{\tau}_h)_{\mathrm{div},h} &= 0 && \forall \underline{\tau}_h \in \underline{X}^k_{\mathrm{curl},h},\\
(\underline{C}^k_h \underline{H}_h, \underline{v}_h)_{\mathrm{div},h} + (\underline{D}^k_h \underline{A}_h, \underline{D}^k_h \underline{v}_h)_{\mathrm{L},h} &= (\underline{I}^k_{\mathrm{div},h} J, \underline{v}_h)_{\mathrm{div},h} && \forall \underline{v}_h \in \underline{X}^k_{\mathrm{div},h}
\end{aligned}
$$

- **Stability follows mimicking the continuous argument for well-posedness**

Restoring function spaces

- The above construction is fully discrete. Can we restore function spaces?

$$
\begin{array}{cccccccc}
 & H^1(\Omega) & \xrightarrow{\text{grad}} & H(\text{curl};\Omega) & \xrightarrow{\text{curl}} & H(\text{div};\Omega) & \xrightarrow{\text{div}} & L^2(\Omega) \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\text{VEM} & V^k_{\text{grad},h} & \xrightarrow{\text{grad}} & V^k_{\text{curl},h} & \xrightarrow{\text{curl}} & V^k_{\text{div},h} & \xrightarrow{\text{div}} & \mathcal{P}^{k-1}(\mathcal{T}_h) \\
 & {\scriptstyle \text{dofs}^{-1}\,\cong}\uparrow & & {\scriptstyle \text{dofs}^{-1}\,\cong}\uparrow & & {\scriptstyle \text{dofs}^{-1}\,\cong}\uparrow & & \updownarrow \\
\text{DDR} & \underline{X}^k_{\text{grad},h} & \xrightarrow{\underline{G}^k_h} & \underline{X}^k_{\text{curl},h} & \xrightarrow{\underline{C}^k_h} & \underline{X}^k_{\text{div},h} & \xrightarrow{D^k_h} & \mathcal{P}^{k-1}(\mathcal{T}_h)
\end{array}
$$

- The spaces $V^k_{\bullet,h}$ are finite-dimensional but not polynomial in general
- This is the key idea of the Virtual Element Method (VEM)[6]

[6] See [Beirão da Veiga et al., 2013] and [Beirão da Veiga et al., 2016 and 2018a] for complexes, [Beirão da Veiga, Dassi, DP, and Droniou, 2022] for the present construction

An example: The virtual curl space I

- For $X \in \mathcal{T}_h \cup \mathcal{F}_h$, let $\mathcal{P}^{k-1|k+1}(X)$ be s.t. $\mathcal{P}^{k+1}(X) = \mathcal{P}^{k-1}(X) \oplus \mathcal{P}^{k-1|k+1}(X)$
- The curl space on a face $F \in \mathcal{F}_h$ is

$$V_{\rm curl}^k(F) := \Big\{ v \in L^2(F)^{d-1} \;:\; {\rm div}_F\, v \in \mathcal{P}^{k+1}(F),\ {\rm rot}_F\, v \in \mathcal{P}^k(F),\\ v \cdot t_E \in \mathcal{P}^k(E) \text{ for all } E \in \mathcal{E}_F,\\ \int_F (v - \pi_{\mathcal{P}^k(F)} v) \cdot (x - x_F)\, p = 0 \text{ for all } p \in \mathcal{P}^{k-1|k+1}(F) \Big\}$$

- The curl space on a mesh element $T \in \mathcal{T}_h$ is

$$V_{\rm curl}^k(T) := \Big\{ v \in L^2(T)^d \;:\; n_{TF} \times (v \times n_{TF}) \in V_{\rm curl}^k(F) \text{ for all } F \in \mathcal{F}_T,\\ \int_T ({\rm curl}\, v - \pi_{\mathcal{P}^k(T)^d}\, {\rm curl}\, v) \cdot (x_T \times w) = 0 \text{ for all } w \in \mathcal{P}^{k-1|k}(T)^d,\\ \int_T (v - \pi_{\mathcal{P}^k(T)^d} v) \cdot (x - x_T)\, p = 0 \text{ for all } p \in \mathcal{P}^{k-1|k+1}(T) \Big\}$$

An example: The virtual curl space II

- The global curl space is defined setting

$$V^k_{\text{curl}}(\mathcal{T}_h) := \left\{ v \in H(\text{curl};\Omega) \,:\, v_{|T} \in V^k_{\text{curl}}(T) \text{ for all } T \in \mathcal{T}_h \right\}$$

- The degrees of freedom are:
  - For each edge $E \in \mathcal{E}_h$,

$$V^k_{\text{curl}}(\mathcal{T}_h) \ni v \mapsto \int_E (v \cdot t_E)\, p \in \mathbb{R} \quad \forall p \in \mathcal{P}^k(E)$$

  - If $k \geq 1$, for each face $F \in \mathcal{F}_h$,

$$V^k_{\text{curl}}(\mathcal{T}_h) \ni v \mapsto \int_F n_F \times (v \times n_F) \cdot w \in \mathbb{R} \quad \forall w \in \mathcal{RT}^k(F)$$

  - If $k \geq 1$, for each element $T \in \mathcal{T}_h$,

$$V^k_{\text{curl}}(\mathcal{T}_h) \ni v \mapsto \int_T v \cdot w \in \mathbb{R} \quad \forall w \in \mathcal{RT}^k(T)$$

Extension to differential forms I

- Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a polytopal domain or manifold
- FE have been extended to the de Rham complex of differential forms[7]:

$$\cdots \longrightarrow H\Lambda^{i-1}(\Omega) \xrightarrow{\mathrm{d}^i} H\Lambda^{i}(\Omega) \longrightarrow \cdots$$

- This has lead to new elements, advanced complexes, etc.
- A Polytopal Exterior Calculus (PEC) framework has been recently presented in [Bonaldi, DP, Droniou and Hu, 2023]

[7] See, e.g., [Bossavit, 1988, Hiptmair, 2002, Arnold et al., 2006]

Outline

Examples of research avenues on polytopal complexes

- Generalization to PEC of relevant analytical results:
  - Poincaré- and Sobolev-type inequalities
  - Adjoint consistency of discrete differential operators
  - ...
- Extension to advanced complexes (e.g., through the BGG construction[8])
  - Stokes complex [Beirão da Veiga et al., 2020, Hanot, 2023]
  - Two-dimensional rot-rot complex [DP, 2023]
  - Three-dimensional div-div complex [DP and Hanot, 2024]
  - ...
- Extension of serendipity techniques to PEC
- Applications of the new complexes to model problems
- **See the other talks of this session for more!**

[8][Arnold and Hu, 2021]

Funded by the European Union (ERC Synergy, NEMESIS, project number 101115663). Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

Thank you for your attention!

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