Analysis approaches for polytopal schemes – the linear and nonlinear cases

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NEMESIS
New generation methods
for numerical simulations

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Research cluster 3: Taming physical complexity

- Tools for incomplete differential operators
  - Development of Discrete Functional Analysis (DFA) for differential operators beyond the gradient
  - Development of general analysis frameworks covering problems involving such operators
  - Extension of the above tools to problems set on manifolds
- Hybrid-dimensional and interface problems
  - Mesh transfer operators and efficient algorithms for moving meshes
  - Application of PEC to systems of PDEs featuring heterogeneous dimensionality
  - Applications to moving domain, contact, and model fluid-structure interaction problems

Outline

1. Linear models: error estimates
   - Stability
   - Consistency
2. Nonlinear analysis

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3rd Strang Lemma

- Continuous space U, discrete space U_h with norm $\|\cdot\|_{\mathsf{U},h}$.
- Continuous problem:

$$\text{Find } u \in \mathsf{U} \text{ such that } a(u,v) = \ell(v) \text{ for all } v \in \mathsf{U}.$$

- Discrete problem:

$$\text{Find } u_h \in \mathsf{U}_h \text{ such that } a_h(u_h,v_h) = \ell_h(v_h) \text{ for all } v_h \in \mathsf{U}_h.$$

- Assume the discrete inf-sup condition

$$\sup_{v_h \in \mathsf{U}_h \setminus \{0\}} \frac{a_h(w_h,v_h)}{\|v_h\|_{\mathsf{U},h}} \geq \alpha \|w_h\|_{\mathsf{U},h} \quad \forall w_h \in \mathsf{U}_h.$$

3rd Strang Lemma

Lemma (3rd Strang Lemma ([1]))

Let $I_h u \in \mathsf{U}_h$ be an "interpolate" of the continuous solution in the discrete space. Then,

$$\|I_h u - u_h\|_{\mathsf{U},h} \leq \alpha^{-1} \sup_{v_h \in \mathsf{U}_h \setminus \{0\}} \frac{\mathcal{E}_h(u; v_h)}{\|v_h\|_{\mathsf{U},h}}$$

where the consistency error is

$$\mathcal{E}_h(u; v_h) = \ell_h(v_h) - a_h(I_h u, v_h) \quad \forall v_h \in \mathsf{U}_h.$$

- Under uniform continuity of a_h, this is actually $\simeq$.

[1] [Di Pietro and Droniou, 2018]; see also [Cangiani et al., 2017] for VEM

Model problem: Stokes in curl-curl formulation

- Ω bounded (polytopal) domain, $\boldsymbol{f} \in L^2(\Omega)^d$, $\nu > 0$.
- Find $(\boldsymbol{u}, p)$ s.t. ([2])

$$\begin{aligned} \nu\, \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{u} + \mathbf{grad}\, p = \boldsymbol{f} &\quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 &\quad \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0 \text{ and } p = 0 &\quad \text{on } \partial\Omega. \end{aligned}$$

- Weak form: Find $(\boldsymbol{u}, p) \in \boldsymbol{H}_0(\mathbf{curl}; \Omega) \times H^1_0(\Omega)$ s.t.

$$\begin{aligned} \nu(\mathbf{curl}\, \boldsymbol{u}, \mathbf{curl}\, \boldsymbol{v})_\Omega + (\mathbf{grad}\, p, \boldsymbol{v})_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega &\quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega), \\ -(\boldsymbol{u}, \mathbf{grad}\, q)_\Omega = 0 &\quad \forall q \in H^1_0(\Omega). \end{aligned}$$

[2] [Girault, 1990]

Discrete setup and scheme

- Discrete spaces and operators, in a complex.

$$H_0^1(\Omega) \xrightarrow{\ \mathbf{grad}\ } \boldsymbol{H}_0(\mathbf{curl};\Omega) \xrightarrow{\ \mathbf{curl}\ } \boldsymbol{H}_0(\mathrm{div};\Omega)$$

$$P_h \xrightarrow{\quad \mathbf{G}_h \quad} \boldsymbol{U}_h \xrightarrow{\quad \mathbf{C}_h \quad} \boldsymbol{Z}_h$$

- L^2-like inner products $(\cdot,\cdot)_{\boldsymbol{U},h}$ and $(\cdot,\cdot)_{\boldsymbol{Z},h}$ on the discrete spaces $\boldsymbol{U}_h$, $\boldsymbol{Z}_h$.
- Approximation $\langle \boldsymbol{f}_h, \cdot\rangle : \boldsymbol{U}_h \to \mathbb{R}$ of $(\boldsymbol{f},\cdot)_\Omega$.
- Scheme ([3]): Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{U}_h \times P_h$ s.t.

$$\begin{aligned} \nu(\mathbf{C}_h\,\boldsymbol{u}_h, \mathbf{C}_h\,\boldsymbol{v}_h)_{\boldsymbol{Z},h} + (\mathbf{G}_h\,p_h, \boldsymbol{v}_h)_{\boldsymbol{U},h} &= \langle \boldsymbol{f}_h, \boldsymbol{v}_h\rangle \quad \forall \boldsymbol{v}_h \in \boldsymbol{U}_h,\\ -(\boldsymbol{u}_h, \mathbf{G}_h\,q_h)_{\boldsymbol{U},h} &= 0 \quad \forall q_h \in P_h. \end{aligned}$$

[3] [Beirão da Veiga et al., 2022a, Di Pietro et al., 2024]

Outline

Inf-sup condition I

Scheme: $a_h((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) = \langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle$ with

$$a_h((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) = \nu(\mathbf{C}_h\,\boldsymbol{u}_h, \mathbf{C}_h\,\boldsymbol{v}_h)_{\boldsymbol{Z},h} + (\mathbf{G}_h\,p_h, \boldsymbol{v}_h)_{\boldsymbol{U},h} - (\boldsymbol{u}_h, \mathbf{G}_h\,q_h)_{\boldsymbol{U},h}.$$

Let

$$\mathcal{S} = \sup_{(\boldsymbol{v}_h, q_h)} \frac{a_h((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h))}{|||(\boldsymbol{v}_h, q_h)|||_h}$$

with $|||(\boldsymbol{v}_h, q_h)|||_h := \|\boldsymbol{v}_h\|_{\boldsymbol{U},h} + \|\,\mathbf{C}_h\,\boldsymbol{v}_h\|_{\boldsymbol{Z},h} + \|\,\mathbf{G}_h\,q_h\|_{\boldsymbol{U},h}$.

- Make $\boldsymbol{v}_h = \boldsymbol{u}_h + \mathbf{G}_h\,p_h$, $q_h = p_h$ and use the

Discrete complex property $\mathbf{C}_h \circ \mathbf{G}_h = 0$

$$\rightsquigarrow \nu\|\,\mathbf{C}_h\,\boldsymbol{u}_h\|^2_{\boldsymbol{Z},h} + \|\,\mathbf{G}_h\,p_h\|^2_{\boldsymbol{U},h} \leq \mathcal{S}|||(\boldsymbol{u}_h, p_h)|||_h.$$

Inf-sup condition II

Reminders:

$$a_h((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) = \nu(\mathbf{C}_h\, \boldsymbol{u}_h, \mathbf{C}_h\, \boldsymbol{v}_h)_{\boldsymbol{Z},h} + (\mathbf{G}_h\, p_h, \boldsymbol{v}_h)_{\boldsymbol{U},h} - (\boldsymbol{u}_h, \mathbf{G}_h\, q_h)_{\boldsymbol{U},h}$$

$$\nu\| \mathbf{C}_h\, \boldsymbol{u}_h\|^2_{\boldsymbol{Z},h} + \| \mathbf{G}_h\, p_h\|^2_{\boldsymbol{U},h} \leq \mathcal{S}|\!|\!|(\boldsymbol{u}_h, p_h)|\!|\!|_h.$$

- Decompose $\boldsymbol{u}_h = \boldsymbol{u}_h^{\perp} + \boldsymbol{u}_h^{*} \in (\operatorname{Ker}\mathbf{C}_h)^{\perp} \oplus \operatorname{Ker}\mathbf{C}_h$.
- Use the:

> Discrete Poincaré inequality: $\|\boldsymbol{v}_h\|_{\boldsymbol{U},h} \lesssim \| \mathbf{C}_h\, \boldsymbol{v}_h\|_{\boldsymbol{Z},h}$ for all $\boldsymbol{v}_h \in (\operatorname{Ker}\mathbf{C}_h)^{\perp}$.

to get

$$\|\boldsymbol{u}_h^{\perp}\|^2_{\boldsymbol{U},h} \lesssim \| \mathbf{C}_h\, \boldsymbol{u}_h^{\perp}\|^2_{\boldsymbol{Z},h} = \| \mathbf{C}_h\, \boldsymbol{u}_h\|^2_{\boldsymbol{Z},h} \lesssim \mathcal{S}|\!|\!|(\boldsymbol{u}_h, p_h)|\!|\!|_h.$$

Inf-sup condition III

Reminders:

$$a_h((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) = \nu(\mathbf{C}_h\, \boldsymbol{u}_h, \mathbf{C}_h\, \boldsymbol{v}_h)_{\boldsymbol{Z},h} + (\mathbf{G}_h\, p_h, \boldsymbol{v}_h)_{\boldsymbol{U},h} - (\boldsymbol{u}_h, \mathbf{G}_h\, q_h)_{\boldsymbol{U},h}$$

$$\nu\|\, \mathbf{C}_h\, \boldsymbol{u}_h\|^2_{\boldsymbol{Z},h} + \|\, \mathbf{G}_h\, p_h\|^2_{\boldsymbol{U},h} + \|\boldsymbol{u}_h^{\perp}\|^2_{\boldsymbol{U},h} \le \mathcal{S} |\!|\!| (\boldsymbol{u}_h, p_h) |\!|\!|_h.$$

with $\boldsymbol{u}_h = \boldsymbol{u}_h^{\perp} + \boldsymbol{u}_h^{*} \in (\mathrm{Ker}\, \mathbf{C}_h)^{\perp} \oplus \mathrm{Ker}\, \mathbf{C}_h$.

- Assuming Ω has a trivial topology, we have $\boldsymbol{u}_h^{*} = \mathbf{G}_h\, r_h$ thanks to the

Exactness of the discrete complex $\mathrm{Ker}\, \mathbf{C}_h = \mathrm{Im}\, \mathbf{G}_h$

- Plug $(\boldsymbol{v}_h, q_h) = (0, r_h)$ and use the Young inequality:

$$\|\boldsymbol{u}_h^{*}\|^2_{\boldsymbol{U},h} \lesssim \mathcal{S} |\!|\!| (\boldsymbol{u}_h, p_h) |\!|\!|_h.$$

Inf-sup condition IV

$$P_h \xrightarrow{\mathbf{G}_h} \boldsymbol{U}_h \xrightarrow{\mathbf{C}_h} \boldsymbol{Z}_h$$

Take-home message: For stability, the discrete sequence must:

- be a complex (discrete calculus relations),
- respect the cohomology (e.g. exactness) of the continuous complex,
- satisfy uniform Poincaré inequalities for operators in the complex.

Examples:

- Finite Element Exterior Calculus [Arnold et al., 2006, Arnold, 2018, Arnold and Hu, 2021].
- Virtual Element Method [Beirão da Veiga et al., 2018, Beirão da Veiga et al., 2021].
- Discrete De Rham method [Di Pietro et al., 2020, Di Pietro and Droniou, 2021b, Di Pietro and Droniou, 2023, Di Pietro and Hanot, 2024].

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Splitting the consistency error

- The consistency error is (I_h interpolators in the proper space):

$$\mathcal{E}_h(u; v_h) = \langle \boldsymbol{f}_h, \boldsymbol{v}_h \rangle \\ - \left[\nu (\mathbf{C}_h (I_h \boldsymbol{u}), \mathbf{C}_h \boldsymbol{v}_h)_{\boldsymbol{Z},h} + (\mathbf{G}_h (I_h p), \boldsymbol{v}_h)_{\boldsymbol{U},h} - (I_h \boldsymbol{u}, \mathbf{G}_h q_h)_{\boldsymbol{U},h} \right].$$

- Recall that $\boldsymbol{f} = \nu \, \mathbf{curl} \, \mathbf{curl} \, \boldsymbol{u} + \mathbf{grad} \, p$ and split the consistency error:

$$\begin{aligned} \mathcal{E}_h(u; v_h) = \nu \big[& \langle (\mathbf{curl} \, \mathbf{curl} \, \boldsymbol{u})_h, \boldsymbol{v}_h \rangle - ((\mathbf{C}_h \circ I_h) \boldsymbol{u}, \mathbf{C}_h \boldsymbol{v}_h)_{\boldsymbol{Z},h} \big] \\ & + \left[\langle (\mathbf{grad} \, p)_h, \boldsymbol{v}_h \rangle - ((\mathbf{G}_h \circ I_h) p, \boldsymbol{v}_h)_{\boldsymbol{U},h} \right] \\ & - (I_h \boldsymbol{u}, \mathbf{G}_h q_h)_{\boldsymbol{U},h}. \end{aligned}$$

Primal consistency

$$\begin{aligned}\mathcal{E}_h(u;v_h) = \nu\Big[&\langle(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u})_h, \boldsymbol{v}_h\rangle - ((\mathbf{C}_h \circ I_h)\boldsymbol{u}, \mathbf{C}_h\,\boldsymbol{v}_h)_{\boldsymbol{Z},h}\Big]\\ &+ \Big[\langle(\mathbf{grad}\,p)_h, \boldsymbol{v}_h\rangle - ((\mathbf{G}_h \circ I_h)p, \boldsymbol{v}_h)_{\boldsymbol{U},h}\Big]\\ &- (I_h\boldsymbol{u}, \mathbf{G}_h\,q_h)_{\boldsymbol{U},h}.\end{aligned}$$

With $\mathcal{D}$ continuous operator, $\mathcal{D}_h$ discrete operator and I_h interpolator:

$\mathcal{D}_h \circ I_h$ approximates $\mathcal{D}$.

- *Example:* $\Big[\langle(\mathbf{grad}\,p)_h, \boldsymbol{v}_h\rangle - ((\mathbf{G}_h \circ I_h)p, \boldsymbol{v}_h)_{\boldsymbol{U},h}\Big] \leq C(p)h^{\ell}\|\boldsymbol{v}_h\|_{\boldsymbol{U},h}$.
- *Straightforward to prove.*

Adjoint consistency

$$\begin{aligned}\mathcal{E}_h(u;v_h) = \nu\Big[\langle(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u})_h, \boldsymbol{v}_h\rangle - ((\mathbf{C}_h \circ I_h)\boldsymbol{u}, \mathbf{C}_h\, \boldsymbol{v}_h)_{\boldsymbol{Z},h}\Big] \\ + \Big[\langle(\mathbf{grad}\, p)_h, \boldsymbol{v}_h\rangle - ((\mathbf{G}_h \circ I_h)p, \boldsymbol{v}_h)_{\boldsymbol{U},h}\Big] \\ - (I_h\boldsymbol{u}, \mathbf{G}_h\, q_h)_{\boldsymbol{U},h}.\end{aligned}$$

Approximate discrete integration by parts involving $(\cdot)_h$, discrete operators and interpolators.

- *Examples:* $(I_h\boldsymbol{u}, \mathbf{G}_h\, q_h)_{\boldsymbol{U},h} + (I_h \underbrace{\mathrm{div}\, \boldsymbol{u}}_{0}, q_h)_{P,h} \lesssim C(\boldsymbol{u})h^\ell \|\mathbf{G}_h\, q_h\|_{\boldsymbol{U},h}$,

$$\Big[\langle(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u})_h, \boldsymbol{v}_h\rangle - ((\underbrace{\mathbf{C}_h \circ I_h}_{I_h(\mathbf{curl}\,\boldsymbol{u})})\boldsymbol{u}, \mathbf{C}_h\, \boldsymbol{v}_h)_{\boldsymbol{Z},h}\Big] \lesssim C(\boldsymbol{u})h^\ell \left(\|\boldsymbol{v}_h\|_{\boldsymbol{U},h} + \|\mathbf{C}_h\, \boldsymbol{v}_h\|_{\boldsymbol{Z},h}\right).$$

- *Can be very challenging to establish!*

Adjoint consistency

Question: w in functional space, v_h discrete vector,

$$\langle (\mathcal{D}^\star w)_h, v_h \rangle - (I_h w, \mathcal{D}_h v_h)_h \lesssim C(w) h^\ell \|v_h\|_h.$$

Adjoint consistency
Fully discrete approach

- Idea: Introduce a suitable (local) polynomial functions $(z_T)_{T\in\mathcal{T}_h}$ approximating w and valid in the definition of $\mathcal{D}_h$.
- Challenge: estimate a quantity

$$\langle(\mathcal{D}^\star w)_h, v_h\rangle - (I_h w, \mathcal{D}_h v_h)_h$$
$$= \sum_{T\in\mathcal{T}_h}\int_T B_T(z_T - w, v_h) + \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_h}\int_F \mathrm{tr}(z_T - w)\gamma_F v_h.$$

$$\sum_{T\in\mathcal{T}_h}\int_T B_T(z_T - w, v_h) + \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_h}\int_F \mathrm{tr}(z_T - w)\gamma_F v_h$$

  - Optimal approximation of $B_T(z_T - w, v_h)$ straightforward.
  - Reduced approximation properties for *traces* of $z_T - w$.

Adjoint consistency

Fully discrete approach

- Challenge: estimate a quantity

$$\sum_{T\in\mathcal{T}_h}\int_T B_T(z_T - w, v_h) + \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_h}\int_F \mathrm{tr}(z_T - w)\gamma_F v_h$$

- Two easy cases:
  - $\boxed{\mathcal{D} = \mathbf{grad}:}$ integrate by parts some volumetric terms to get

$$\sum_{T\in\mathcal{T}_h}\int_T \tilde{B}_T(z_T - w, v_h) + \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_h}\int_F \mathrm{tr}(z_T - w)(\gamma_F v_h - P_T v_h).$$

    and use the properties of the (discrete) gradient to get

$$\|\gamma_F v_h - P_T v_h\|_{L^2(F)} \leq h_T \|\,\mathbf{grad}_T\, v_h\|_T.$$

Adjoint consistency

Fully discrete approach

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    and use the properties of the (discrete) gradient to get

$$\|\gamma_F v_h-P_T v_h\|_{L^2(F)}\leq h_T\|\mathbf{grad}_T\,v_h\|_T.$$

  - $\boxed{\mathcal{D}=\mathrm{div}:}$ degree of z_T large enough $\rightsquigarrow$ optimal estimate on $\|\mathrm{tr}(z_T-w)\|_{L^2(F)}$.

Adjoint consistency

Fully discrete approach

- Challenge: estimate a quantity

$$\sum_{T\in\mathcal{T}_h}\int_T B_T(z_T - w, v_h) + \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_h}\int_F \mathrm{tr}(z_T - w)\gamma_F v_h$$

- $\boxed{\mathcal{D} = \mathbf{curl}}$ is the difficult case...
  - Introduce an $\boldsymbol{H}(\mathbf{curl};\Omega)$ conforming Rv_h whose suitable projection on F matches $\gamma_F v_h$, then locally integrate by parts:

$$\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_h}\int_F \mathrm{tr}(z_T - w)\gamma_F v_h \rightsquigarrow \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_h}\int_F \mathrm{tr}(z_T - w)Rv_h$$

$$\overset{IBP}{=} \sum_{T\in\mathcal{T}_h}\int_T \widehat{B}_T(z_T - w, v_h).$$

Constructing Rv_h requires to locally solve $\mathbf{curl}$–div problems and use fine PDE estimates ([a]).

[a] [Di Pietro and Droniou, 2021a]

Adjoint consistency

Virtual functions approach

- Idea: use the conformity of v_h to integrate by parts, and introduce a (local) polynomial function...
- After straightforward approximation properties of source term:

$$\mathcal{T} = \int_\Omega \mathcal{D}^\star w \, v_h - (I_h w, \mathcal{D} v_h)_h.$$

Issue: $(\cdot,\cdot)_h$ is a *discrete, non-conforming* L^2 inner product, IBP cannot be directly used.

- After straightforward approximation properties of source term:

$$\mathcal{T} = \int_\Omega \mathcal{D}^\star w\, v_h - (I_h w, \mathcal{D} v_h)_h.$$

- Introduce a piecewise polynomial function z_h, approximation of w, and use (primal) consistency of $(\cdot,\cdot)_h$:

$$\begin{aligned}\mathcal{T} &= \int_\Omega \mathcal{D}^\star w\, v_h - \int_\Omega z_h \mathcal{D} v_h + (I_h w - z_h, \mathcal{D} v_h)_h \\ &= \underbrace{\int_\Omega \mathcal{D}^\star w\, v_h - \int_\Omega w \mathcal{D} v_h}_{=0} + \int_\Omega \underbrace{(z_h - w)}_{\text{easy}} \mathcal{D} v_h + (I_h w - z_h, \mathcal{D} v_h)_h.\end{aligned}$$

Adjoint consistency

Virtual functions approach

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  - Write $I_h w - z_h = (I_h w - w) + (w - z_h)$: we need to estimate the interpolation error $I_h w - w$ on the virtual (not polynomial) space.

> Estimating $I_h w - w$ requires stability of I_h, based on fine PDE estimates ([a]) on systems involving **curl**, div, **grad**.
>
> [a] [Beirão da Veiga et al., 2022b]

Outline

When can we get error estimates?

- *Remark 1*: estimate of consistency error in $\mathcal{O}(h^\ell)$ requires some smoothness of the solution.
- *Remark 2*: error estimates imply uniqueness of the solution.

Proposition

Let u be a solution to the continuous model, and assume that it satisfies Assumption (A). If the following holds:

> *There exists meshes such that, with u_h solution to the scheme, for some norm $\|\cdot\|$, we have $\|u_h - u\| \to 0$ as $h \to 0$,*

then, under Assumption (A), the continuous model has a unique solution.

When can we get error estimates?

- Smoothness: even for a simple (linear) Darcy flow, if the permeability is discontinuous the solution may not belong to $H^2(\mathcal{T}_h)$.
- Uniqueness:
  - *Navier-Stokes: requires smallness of data or strong smoothness assumption on the solution.*
  - *Multiphase flows in porous media, flows in fractured media, etc.: ??*

Alternative: convergence by compactness

- Convergence by compactness: $(u_h)_h$ solutions to the scheme.
  - Prove that $(u_h)_h$ is bounded in a certain (strong) norm.
  - Use this bound to prove that $(u_h)_h$ and $(\mathcal{D}_h u_h)_h$ converge up to a subsequence to u and $\mathcal{D}u$ in a suitable sense (typically, strong on $(u_h)_h$, weak on $(\mathcal{D}_h u_h)_h$),
  - Pass to the limit to see that u solves the continuous problem.

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  - Pass to the limit to see that u solves the continuous problem.
- Strong compactness results (continuous case): Ascoli-Arzela, Kolmogorov, Rellich, Aubin-Simon...
- Adapted to the discrete setting when we can bound the discrete gradient of u_h [Eymard et al., 2000, Di Pietro and Ern, 2010, Li et al., 2015, Droniou et al., 2018].

Alternative: convergence by compactness

- Compactness results still limited for models based on $\mathbf{curl}$ and div.

  Either finite element complexes, or non-complex polytopal methods [Kikuchi, 1989, Boffi, 2001, Lemaire and Pitassi, 2024].

- Even worse when (discrete) Sobolev embeddings are required for the curl [Amrouche et al., 1998, Girault, 1990].

Conclusion

- Properties to establish for error estimates (stability & consistency):
  - Discrete complex with the same cohomology as the continuous one,
  - Poincaré inequalities (for all operators),
  - Primal and adjoint consistency properties.
- Some are easy, others much more challenging (and ongoing).
- For nonlinear models: compactness results are still lacking (a lot).

Ongoing/future questions

- Can we have, as we do for the gradient, a generic framework of discrete functional analysis tools for the curl/divergence?

  Such a framework gives Poincaré, Sobolev, primal and adjoint consistency. [Droniou et al., 2018]

- Discrete complexes (and analysis) on manifolds? [Droniou et al., 2024b]

- Discrete complexes (whether finite-element based or polytopal) are inherently hybrid-dimensional constructions.

  $\leadsto$ naturally adapted to problems with interfaces and hybrid dimensions such as:

  - contact problems [Wriggers, 2006, Aldakheel et al., 2020],
  - fluid-structure interactions [Beirão da Veiga et al., 2021],
  - flows in fractured media (including, e.g., elastic behaviours) [Martin et al., 2005, Brenner et al., 2018, Droniou et al., 2024a],
  - etc.

  requires efficient handling of meshes...

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