Convection robust elements in Magnetohydrodynamics (RC4)

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NEMESIS

New generation methods
for numerical simulations

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Structure of RC 4

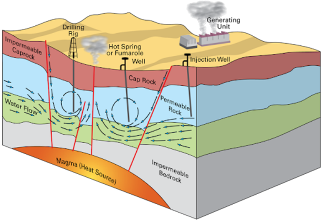
Research Cluster 4 will consider some "proof of concept applications" both from the practical/computational and the theoretical standpoint.

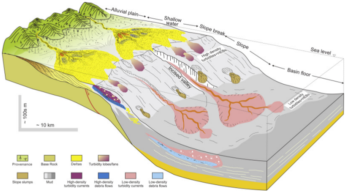
- Geological flows
  - Design and analysis of mixed schemes for multi-phase flows
  - Fractured media
  - Coupling flows and porous media deformations
- Magnetohydrodynamics
  - Design and analysis of robust methods for the incompressible Navier-Stokes and Maxwell equations
  - Coupling and analysis of the full MHD system
  - Efficient resolution of the resulting coupled system of PDEs
  - Particular attention to parameter robustness

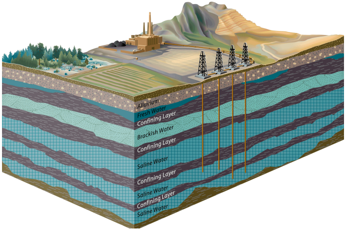
Geological flows

Applications, for instance, in:

- Reservoir simulation
- Basin simulation
- Waste storage, CO^2 sequestration







A glance at the polyhedral literature in G.P.F.⋆

⋆ sorry ... many important names missing!

Some main models/problems are:

- Discrete Fracture Networks
- Flow in fractured (and/or “bad” coefficients) media (matrix and fractures)
- Multi-phase flows, “thermo-aware” models,...
- Solid-mechanics aspects (contact, poro-mechanics, elasto-dynamics, ...)

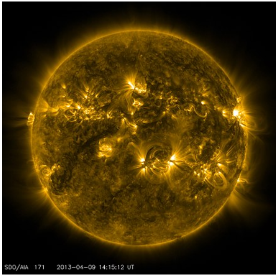
Some main polytopal techs already on-the-field:

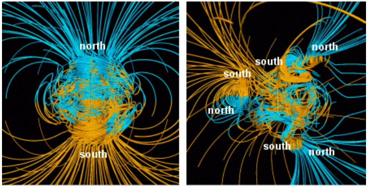
- Mimetic Finite Differences (BdV, Formaggia, Lipnikov,..)
- Gradient Schemes (Bonaldi ,Droniou, Masson, ..)
- Virtual Elements (BdV, Berrone, Brezzi, Dassi, Faille, Masson, ..)
- Hybrid High Order (Botti, Chave, Ern, Di Pietro, ..)
- Polytopal Discontinuous Galerkin (Antonietti, Mazzieri, Verani, ..)
- Discrete De Rham (Di Pietro, Droniou, ..)

Magnetohydrodynamics

<u>Applications, for instance, in:</u>

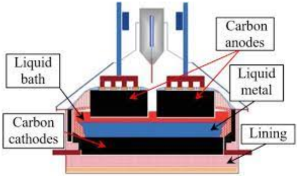
- Space physics
- Geophysics
- Engineering





between reversals

during a reversal



A classical model in MHD (four fields)

Let $\Omega \subset \mathbb{R}^3$. We search for

- $\mathbf{u} : [0, T] \to \mathbb{R}^3$ velocity field; $p : [0, T] \to \mathbb{R}$ pressure field
- $\mathbf{B} : [0, T] \to \mathbb{R}^3$ magnetic 'field' $\mathbf{E} : [0, T] \to \mathbb{R}^3$ electric field

that satisfy the equations (at all admissible times)

$$
\begin{cases}
\rho\, \partial_t \mathbf{u} + \rho(\nabla \mathbf{u})\mathbf{u} - \nu \Delta \mathbf{u} - \mathbf{j} \times \mathbf{B} + \nabla p = \mathbf{f} & \text{in } \Omega, \\
\mathbf{j} - \mu^{-1} \mathbf{curl}\, \mathbf{B} = \mathbf{0} & \text{in } \Omega, \\
\partial_t \mathbf{B} + \mathbf{curl}\, \mathbf{E} = \mathbf{0} & \text{in } \Omega, \\
\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) & \text{in } \Omega, \\
\mathrm{div}\, \mathbf{B} = 0\,,\, \mathrm{div}\, \mathbf{u} = 0 & \text{in } \Omega,
\end{cases}
$$

coupled with initial conditions (on **u** and **B**) and boundary conditions, e.g.

$$
\mathbf{u} = 0\,, \quad \mathbf{B} \cdot \mathbf{n} = 0\,, \quad \mathbf{E} \times \mathbf{n} = 0 \quad \text{on } \partial\Omega\,.
$$

A classical model in MHD (three fields formulation)

By eliminating the electric field, one obtains the alternative equations for

- $\mathbf{u} : [0, T] \to \mathbb{R}^3$ velocity field;
- $p : [0, T] \to \mathbb{R}$ pressure field
- $\mathbf{B} : [0, T] \to \mathbb{R}^3$ magnetic 'field' ,

that need to satisfy

$$\begin{cases} \rho\,\partial_t \mathbf{u} + \rho(\nabla\mathbf{u})\mathbf{u} - Re^{-1}\Delta\mathbf{u} + \mu^{-1}\mathbf{B} \times \mathbf{curl}\,\mathbf{B} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \partial_t \mathbf{B} + \mathbf{curl}\,(\sigma\mu)^{-1}\mathbf{curl}\,\mathbf{B} - \mathbf{curl}\,(\mathbf{u} \times \mathbf{B}) = \mathbf{0} & \text{in } \Omega, \\ \operatorname{div}\mathbf{B} = 0\,, \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \end{cases}$$

coupled with initial conditions (on **u** and **B**) and boundary conditions, e.g.

$$\mathbf{u} = 0\,, \quad \mathbf{B} \cdot \mathbf{n} = 0\,, \quad \mathbf{curl}\,\mathbf{B} \times \mathbf{n} = 0 \quad \text{on } \partial\Omega\,.$$

A rich Finite Element literature

A few, among the many, involved names:

L. Chacon, R. Codina, J. Evans, F. Gay-Balmaz, J.-F. Gerbeau, J.L. Guermond, M.D. Gunzburger, Y. He, R. Hiptmair, P. Houston, K. Hu, W. Layton, A. Prohl, D. Shotzau, J. Xu, ...

Setting variety in the literature:

- stationary or time-dependent problem
- different formulations (different fields or potentials)
- regular or non-regular domains ($H_{\mathrm{div}\cap\mathrm{curl}}$ vs. H^1)
- many choices of FEM
- focus on different aspects/difficulties (next slide...)

Some (numerical analyst's) challenges

Many different aspects are investigated:

- time-stepping choices (implicit/explicit, coupled/uncoupled, ...)
- associated nonlinear solvers (convergence of iterations, costs,...)
- convergence analysis (with order for regular solutions, or for "vanishing discretization" parameters)
- robustness to high Reynolds and associated stabilizations (theory generally only for linearized case)
- conservation of quantities (solenoidal conditions for u and B, magnetic and cross helicities, ...)
- energy stability, preconditioners, ...

Some NEMESIS assets:

Robust for polyhedral meshes, many complexes easily handled, a focus on efficiency (solvers, adaptivity,..).

Variational formulation of the three field equations

We assume a convex domain Ω and constant coefficients.

Find $\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\boldsymbol{H}_0^1(\Omega))$, $p \in L^2(0,T;L_0^2(\Omega))$, $\boldsymbol{B} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\boldsymbol{H}_{\boldsymbol{n}}^1(\Omega))$, such that for a.e. $t \in [0,T]$

$$\begin{cases} \left(\dfrac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v}\right) + \nu_{\mathrm{S}} a^{\mathrm{S}}(\boldsymbol{u},\boldsymbol{v}) + c(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) - d(\boldsymbol{B};\boldsymbol{B},\boldsymbol{v}) + b(\boldsymbol{v},p) = (\boldsymbol{f},\boldsymbol{v}), \\ b(\boldsymbol{u},q) = 0 \qquad\qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega),\ \forall q \in L_0^2(\Omega), \\ \left(\dfrac{\partial \boldsymbol{B}}{\partial t}, \boldsymbol{H}\right) + \nu_{\mathrm{M}} a^{\mathrm{M}}(\boldsymbol{B},\boldsymbol{H}) + d(\boldsymbol{B};\boldsymbol{H},\boldsymbol{u}) = (\boldsymbol{G},\boldsymbol{H}) \qquad \forall \boldsymbol{H} \in \boldsymbol{H}_{\boldsymbol{n}}^1(\Omega), \end{cases}$$

coupled with initial conditions $(\mathrm{div}\,\mathbf{B}(\cdot,0) = 0)$.

$$a^{\mathrm{S}}(\boldsymbol{u},\boldsymbol{v}) = (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v})), \quad c(\chi;\boldsymbol{u},\boldsymbol{v}) = ((\boldsymbol{\nabla}\boldsymbol{u})\,\chi,\ \boldsymbol{v}),$$

$$a^{\mathrm{M}}(\boldsymbol{B},\boldsymbol{H}) = (\mathbf{curl}(\boldsymbol{B}),\ \mathbf{curl}(\boldsymbol{H})) + (\mathrm{div}(\boldsymbol{B}), \mathrm{div}(\boldsymbol{H})),$$

$$b(\boldsymbol{v},q) = (\mathrm{div}\,\boldsymbol{v},\ q), \qquad d(\boldsymbol{\Theta};\boldsymbol{H},\boldsymbol{v}) = (\mathbf{curl}(\boldsymbol{H}) \times \boldsymbol{\Theta},\ \boldsymbol{v}).$$

The discrete problem (type 1)

The following approach was initially proposed in [BdV, Dassi, Vacca, SINUM, 2024] for the (stationary) linearized case and later generalized [ArXiv, and submitted] to the (evolutionary) nonlinear case.

Discrete spaces: $(k \geq 1)$

$$\boldsymbol{V}_k^h = [\mathbb{P}_k(\Omega_h)]^3 \cap \boldsymbol{H}_0(\mathrm{div}) \qquad \text{velocity field}\,,$$
$$Q_k^h = \mathbb{P}_{k-1}(\Omega_h) \cap L_0^2(\Omega) \qquad \text{pressure field}\,,$$
$$\boldsymbol{W}_k^h = [\mathbb{P}_k^{\mathrm{cont}}(\Omega_h)]^3 \cap \boldsymbol{H}_{\boldsymbol{n}}^1(\Omega) \qquad \text{magnetic field}\,.$$

The non-conforming couple $(\boldsymbol{V}_k^h, Q_k^h)$, combined with upwinding, is a very robust choice for incompressible fluids, see for instance [Barrenechea, Burman, Guzman, 2019], [Han, Hou, 2021].

Note that the convexity of the domain allows us to safely use an H^1-conforming space for the discrete magnetic field.

The discrete problem (type 1)

Find $\boldsymbol{u}_h \in L^\infty(0,T;\boldsymbol{V}_k^h)$, $p_h \in L^2(0,T;Q_k^h)$, $\boldsymbol{B}_h \in L^\infty(0,T;\boldsymbol{W}_k^h)$, such that for a.e. $t \in I$

$$\begin{cases} \left(\dfrac{\partial \boldsymbol{u}_h}{\partial t}, \boldsymbol{v}_h\right) + \nu_{\mathrm{S}} a_h^{\mathrm{S}}(\boldsymbol{u}_h, \boldsymbol{v}_h) + c_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) - d(\boldsymbol{B}_h; \boldsymbol{B}_h, \boldsymbol{v}_h) \\ \qquad + J_h(\boldsymbol{B}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \ \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_k^h, \\ b(\boldsymbol{u}_h, q_h) = 0 \ \ \forall q_h \in Q_k^h, \\ \left(\dfrac{\partial \boldsymbol{B}_h}{\partial t}, \boldsymbol{H}_h\right) + \nu_{\mathrm{M}} a^{\mathrm{M}}(\boldsymbol{B}_h, \boldsymbol{H}_h) + d(\boldsymbol{B}_h; \boldsymbol{H}_h, \boldsymbol{u}_h) + \\ \qquad + (\mathrm{div}\, \boldsymbol{B}_h, \mathrm{div}\, \boldsymbol{H}_h) = (\boldsymbol{G}, \boldsymbol{H}_h) \ \ \forall \boldsymbol{H}_h \in \boldsymbol{W}_k^h, \end{cases}$$

coupled with initial conditions.

The discrete problem (type 1)

Find $\boldsymbol{u}_h \in L^\infty(0,T;\boldsymbol{V}_k^h)$, $p_h \in L^2(0,T;Q_k^h)$, $\boldsymbol{B}_h \in L^\infty(0,T;\boldsymbol{W}_k^h)$, such that for a.e. $t \in I$

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coupled with initial conditions.

$$a_h^{\mathrm{S}}(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\varepsilon_h(\boldsymbol{u}_h), \varepsilon_h(\boldsymbol{v}_h)) - \sum_{f \in \Sigma_h} (\{\{ \varepsilon_h(\boldsymbol{u}_h)\boldsymbol{n}_f \}\}_f, [\![\boldsymbol{v}_h]\!]_f)_f + \\ - \sum_{f \in \Sigma_h} ([\![\boldsymbol{u}_h]\!]_f, \{\{ \varepsilon_h(\boldsymbol{v}_h)\boldsymbol{n}_f \}\}_f)_f + \mu_a \sum_{f \in \Sigma_h} h_f^{-1} ([\![\boldsymbol{u}_h]\!]_f, [\![\boldsymbol{v}_h]\!]_f)_f$$

The discrete problem (type 1)

Find $\boldsymbol{u}_h \in L^\infty(0,T;\boldsymbol{V}_k^h)$, $p_h \in L^2(0,T;Q_k^h)$, $\boldsymbol{B}_h \in L^\infty(0,T;\boldsymbol{W}_k^h)$,
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coupled with initial conditions.

$$c_h(\chi; \boldsymbol{u}_h, \boldsymbol{v}_h) = ((\boldsymbol{\nabla}_h \boldsymbol{u}_h)\,\chi,\ \boldsymbol{v}_h) - \sum_{f \in \Sigma_h^{\mathrm{int}}} ((\chi \cdot \boldsymbol{n}_f)[\![\, \boldsymbol{u}_h \,]\!]_f,\ \{\{\, \boldsymbol{v}_h \,\}\}_f)_f +$$
$$+ \mu_c \sum_{f \in \Sigma_h^{\mathrm{int}}} (|\chi \cdot \boldsymbol{n}_f| [\![\, \boldsymbol{u}_h \,]\!]_f,\ [\![\, \boldsymbol{v}_h \,]\!]_f)_f$$

The discrete problem (type 1)

Find $\boldsymbol{u}_h \in L^\infty(0,T;\boldsymbol{V}_k^h)$, $p_h \in L^2(0,T;Q_k^h)$, $\boldsymbol{B}_h \in L^\infty(0,T;\boldsymbol{W}_k^h)$,
such that for a.e. $t \in I$

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coupled with initial conditions.

$$J_h(\boldsymbol{\Theta}; \boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{f \in \Sigma_h^{\mathrm{int}}} \max\{\|\boldsymbol{\Theta}\|^2_{L^\infty(\omega_f)}, 1\} \Big(\mu_{J_1} ([\![\boldsymbol{u}_h]\!]_f, [\![\boldsymbol{v}_h]\!]_f)_f \\ + \mu_{J_2} h_f^2 ([\![\nabla_h \boldsymbol{u}_h]\!]_f, [\![\nabla_h \boldsymbol{v}_h]\!]_f)_f \Big)$$

The discrete problem (type 1)

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coupled with initial conditions.

$(\mathrm{div}\, \boldsymbol{B}_h, \mathrm{div}\, \boldsymbol{H}_h)$ ‘strong’ grad-div stabilization on B_h

(not needed in the linearized case)

Interpolation for the magnetic field

CIP analysis is typically based on suitable orthogonality properties.

In order to avoid a quasi-uniformity mesh assumption and Nitsche imposition of BCs, we introduce $\mathcal{I}_{\boldsymbol{W}} \colon \boldsymbol{W} \to \boldsymbol{W}_k^h$ satisfying

$$\left(\boldsymbol{H} - \mathcal{I}_{\boldsymbol{W}}\boldsymbol{H}, \, \boldsymbol{q}_{k-1}\right) = 0 \qquad \text{for any } \boldsymbol{q}_{k-1} \in [\mathbb{O}_{k-1}(\Omega_h)]^3$$

where

$$\mathbb{O}_{k-1}(\Omega_h) := \mathbb{P}_{k-1}^{\text{cont}}(\Omega_h) \quad \text{for } k > 1, \qquad \mathbb{O}_{k-1}(\Omega_h) := \mathbb{P}_0(\widetilde{\Omega}_h) \quad \text{for } k = 1;$$

plus standard LOCAL approximation estimates in L^2 and H^1.

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where

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plus standard LOCAL approximation estimates in L^2 and H^1.

NOTE: It exists a projection operator $\mathcal{I}_{\mathcal{O}} \colon \mathbb{P}_{k-1}(\Omega_h) \to \mathbb{O}_{k-1}(\Omega_h)$ such that for any $p_{k-1} \in \mathbb{P}_{k-1}(\Omega_h)$ the following holds:

$$\sum_{E \in \Omega_h} h_E \|(I - \mathcal{I}_{\mathcal{O}})p_{k-1}\|_E^2 \lesssim \sum_{f \in \Sigma_h^{\text{int}}} h_f^2 \|[\![\, p_{k-1} \,]\!]_f\|_f^2 \,.$$

Theoretical results (linearized stationary case)

Under standard mesh shape regularity, it holds

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{\mathrm{stab}} + \|\boldsymbol{B}-\boldsymbol{B}_h\|^2_{\mathrm{M}} \lesssim (\Lambda^2_{\mathrm{S}} + \Gamma^2_{\mathrm{S}} + \Gamma^2_{\mathrm{M}}) h^{2k} |\boldsymbol{u}|^2_{k+1,\Omega_h} + (\Lambda^2_{\mathrm{M}} + \Gamma^2_{\mathrm{S}}) h^{2k} |\boldsymbol{B}|^2_{k+1,\Omega_h}$$

where

$$\Lambda^2_{\mathrm{S}} := \max\left\{\sigma_{\mathrm{S}} h^2\,, \|\boldsymbol{\chi}\|_{L^\infty(\Omega)} h\,, \|\boldsymbol{\Theta}\|^2_{L^\infty(\Omega)} h\,, \nu_{\mathrm{S}}(1+\mu_a+\mu_a^{-1})\right\},$$

$$\Lambda^2_{\mathrm{M}} := \max\{\sigma_{\mathrm{M}} h^2\,, \nu_{\mathrm{M}}\}$$

$$\Gamma^2_{\mathrm{S}} := \min\{\sigma_{\mathrm{S}}^{-1} h^2, \nu_{\mathrm{S}}^{-1} h^4\} \|\boldsymbol{\chi}\|^2_{W^{1,\infty}(\Omega_h)} + \sigma_{\mathrm{S}}^{-1} h^2 \|\boldsymbol{\Theta}\|^2_{W^{1,\infty}(\Omega_h)} + h$$

$$\Gamma^2_{\mathrm{M}} := \min\{\sigma_{\mathrm{M}}^{-1} h^2, \nu_{\mathrm{M}}^{-1} h^4\} \|\boldsymbol{\Theta}\|^2_{W^{1,\infty}(\Omega_h)}$$

Theoretical results (linearized stationary case)

Under standard mesh shape regularity, it holds

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{\rm stab}+\|\boldsymbol{B}-\boldsymbol{B}_h\|^2_{\rm M}\lesssim (\Lambda^2_{\rm S}+\Gamma^2_{\rm S}+\Gamma^2_{\rm M})h^{2k}|\boldsymbol{u}|^2_{k+1,\Omega_h}+(\Lambda^2_{\rm M}+\Gamma^2_{\rm S})h^{2k}|\boldsymbol{B}|^2_{k+1,\Omega_h}$$

where

$$\Lambda^2_{\rm S}:=\max\left\{\sigma_{\rm S}h^2\,,\|\boldsymbol{\chi}\|_{L^\infty(\Omega)}h\,,\|\boldsymbol{\Theta}\|^2_{L^\infty(\Omega)}h\,,\nu_{\rm S}(1+\mu_a+\mu_a^{-1})\right\},$$

$$\Lambda^2_{\rm M}:=\max\{\sigma_{\rm M}h^2\,,\nu_{\rm M}\}$$

$$\Gamma^2_{\rm S}:=\min\{\sigma_{\rm S}^{-1}h^2,\,\nu_{\rm S}^{-1}h^4\}\|\boldsymbol{\chi}\|^2_{W^{1,\infty}(\Omega_h)}+\sigma_{\rm S}^{-1}h^2\|\boldsymbol{\Theta}\|^2_{W^{1,\infty}(\Omega_h)}+h$$

$$\Gamma^2_{\rm M}:=\min\{\sigma_{\rm M}^{-1}h^2,\,\nu_{\rm M}^{-1}h^4\}\|\boldsymbol{\Theta}\|^2_{W^{1,\infty}(\Omega_h)}$$

- Also optimal pressure estimates in L^2 hold;
- there hold also convergence results by compactness, in a weaker sense, to non regular solutions.

Theoretical results (nonlinear case, type 1)

Under standard mesh shape regularity, it holds

$$\begin{aligned}
&\|(\boldsymbol{u}-\boldsymbol{u}_h)(\cdot,T)\|^2 + \|(\boldsymbol{B}_h-\boldsymbol{B}_h)(\cdot,T)\|^2 \\
&+ \int_0^T \|(\boldsymbol{u}-\boldsymbol{u}_h)(t)\|_{\mathrm{stab}}^2 + \int_0^T \|(\boldsymbol{B}-\boldsymbol{B}_h)(\cdot,t)\|_{3\mathrm{f}}^2 \\
&\lesssim \left(\Lambda_{\mathrm{stab}}^2 \|\boldsymbol{u}\|^2_{L^2(0,T;\boldsymbol{W}_\infty^{k+1}(\Omega_h))} + \Lambda_{3\mathrm{f}}^2 \|\boldsymbol{B}\|^2_{L^2(0,T;\boldsymbol{H}^{k+1}(\Omega_h))}\right) h^{2k} \\
&+ \left(\|\boldsymbol{u}\|^2_{H^1(0,T;\boldsymbol{H}^{k+1}(\Omega_h))} + \|\boldsymbol{B}\|^2_{H^1(0,T;\boldsymbol{H}^{k+1}(\Omega_h))}\right) h^{2k+2},
\end{aligned}$$

where

$$\Lambda_{\mathrm{stab}}^2 := \max\left\{\nu_{\mathrm{S}}(1+\mu_a+\mu_a^{-1}),\, h(\gamma_{\mathrm{data}}^2+1)\right\}, \qquad \Lambda_{3\mathrm{f}}^2 := (\nu_{\mathrm{M}}+1).$$

Theoretical results (nonlinear case, type 1)

Under standard mesh shape regularity, it holds

$$
\begin{aligned}
&\|(\boldsymbol{u}-\boldsymbol{u}_h)(\cdot,T)\|^2+\|(\boldsymbol{B}_h-\boldsymbol{B}_h)(\cdot,T)\|^2\\
&+\int_0^T\|(\boldsymbol{u}-\boldsymbol{u}_h)(t)\|^2_{\mathrm{stab}}+\int_0^T\|(\boldsymbol{B}-\boldsymbol{B}_h)(\cdot,t)\|^2_{3\mathrm{f}}\\
&\lesssim\left(\Lambda^2_{\mathrm{stab}}\|\boldsymbol{u}\|^2_{L^2(0,T;\boldsymbol{W}^{k+1}_\infty(\Omega_h))}+\Lambda^2_{3\mathrm{f}}\|\boldsymbol{B}\|^2_{L^2(0,T;\boldsymbol{H}^{k+1}(\Omega_h))}\right)h^{2k}\\
&+\left(\|\boldsymbol{u}\|^2_{H^1(0,T;\boldsymbol{H}^{k+1}(\Omega_h))}+\|\boldsymbol{B}\|^2_{H^1(0,T;\boldsymbol{H}^{k+1}(\Omega_h))}\right)h^{2k+2}\,,
\end{aligned}
$$

where

$$
\Lambda^2_{\mathrm{stab}}:=\max\left\{\nu_{\mathrm{S}}(1+\mu_a+\mu_a^{-1}),\,h(\gamma^2_{\mathrm{data}}+1)\right\}\,,\qquad \Lambda^2_{3\mathrm{f}}:=(\nu_{\mathrm{M}}+1)\,.
$$

- Quasi-robust and pressure-robust;
- lacks the $O(h^{k+1/2})$ pre-asymptotic error reduction in convective regimes (responsible identified: solenoidal B_h condition)

The discrete problem (type 2)

Find $\boldsymbol{u}_h \in L^\infty(0,T;\boldsymbol{V}_k^h)$, $p_h \in L^2(0,T;Q_k^h)$, $\boldsymbol{B}_h \in L^\infty(0,T;\boldsymbol{W}_k^h)$, $\varphi_h \in L^\infty(0,T;R_k^h)$, such that for a.e. $t \in I$

$$\begin{cases} \left(\dfrac{\partial \boldsymbol{u}_h}{\partial t}, \boldsymbol{v}_h\right) + \nu_{\mathrm{S}} a_h^{\mathrm{S}}(\boldsymbol{u}_h, \boldsymbol{v}_h) + c_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) - d(\boldsymbol{B}_h; \boldsymbol{B}_h, \boldsymbol{v}_h) \\ \qquad + J_h(\boldsymbol{B}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \ \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_k^h, \\ b(\boldsymbol{u}_h, q_h) = 0 \ \ \forall q_h \in Q_k^h, \\ \left(\dfrac{\partial \boldsymbol{B}_h}{\partial t}, \boldsymbol{H}_h\right) + \nu_{\mathrm{M}} a^{\mathrm{M}}(\boldsymbol{B}_h, \boldsymbol{H}_h) + d(\boldsymbol{B}_h; \boldsymbol{H}_h, \boldsymbol{u}_h) + \\ \qquad + K_h(\boldsymbol{u}_h; \boldsymbol{B}_h, \boldsymbol{H}_h) - b(\boldsymbol{H}_h, \varphi_h) = (\boldsymbol{G}, \boldsymbol{H}_h) \ \ \forall \boldsymbol{H}_h \in \boldsymbol{W}_k^h, \\ Y_h(\varphi_h, \psi_h) + b(\boldsymbol{B}_h, \psi_h) = 0 \ \ \forall \psi_h \in R_k^h, \end{cases}$$

$$R_k^h = \mathbb{P}_k^{\mathrm{cont}}(\Omega_h) \cap L_0^2(\Omega)$$

The discrete problem (type 2)

Find $\boldsymbol{u}_h \in L^\infty(0,T;\boldsymbol{V}_k^h)$, $p_h \in L^2(0,T;Q_k^h)$, $\boldsymbol{B}_h \in L^\infty(0,T;\boldsymbol{W}_k^h)$, $\varphi_h \in L^\infty(0,T;R_k^h)$, such that for a.e. $t \in I$

$$
\begin{cases}
\left(\dfrac{\partial \boldsymbol{u}_h}{\partial t}, \boldsymbol{v}_h\right) + \nu_{\mathrm{S}} a_h^{\mathrm{S}}(\boldsymbol{u}_h, \boldsymbol{v}_h) + c_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) - d(\boldsymbol{B}_h; \boldsymbol{B}_h, \boldsymbol{v}_h) \\
\qquad + J_h(\boldsymbol{B}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \;\; \forall \boldsymbol{v}_h \in \boldsymbol{V}_k^h, \\
b(\boldsymbol{u}_h, q_h) = 0 \;\; \forall q_h \in Q_k^h, \\
\left(\dfrac{\partial \boldsymbol{B}_h}{\partial t}, \boldsymbol{H}_h\right) + \nu_{\mathrm{M}} a^{\mathrm{M}}(\boldsymbol{B}_h, \boldsymbol{H}_h) + d(\boldsymbol{B}_h; \boldsymbol{H}_h, \boldsymbol{u}_h) + \\
\qquad + K_h(\boldsymbol{u}_h; \boldsymbol{B}_h, \boldsymbol{H}_h) - b(\boldsymbol{H}_h, \varphi_h) = (\boldsymbol{G}, \boldsymbol{H}_h) \;\; \forall \boldsymbol{H}_h \in \boldsymbol{W}_k^h, \\
Y_h(\varphi_h, \psi_h) + b(\boldsymbol{B}_h, \psi_h) = 0 \;\; \forall \psi_h \in R_k^h,
\end{cases}
$$

$$
Y_h(\varphi_h, \psi_h) = \mu_Y \sum_{f \in \Sigma_h^{\mathrm{int}}} h_f^2 ([\![\nabla \varphi_h]\!]_f, [\![\nabla \psi_h]\!]_f)_f \, .
$$

The discrete problem (type 2)

Find $\boldsymbol{u}_h \in L^\infty(0,T;\boldsymbol{V}_k^h)$, $p_h \in L^2(0,T;Q_k^h)$, $\boldsymbol{B}_h \in L^\infty(0,T;\boldsymbol{W}_k^h)$, $\varphi_h \in L^\infty(0,T;R_k^h)$, such that for a.e. $t \in I$

$$
\begin{cases}
\left(\dfrac{\partial \boldsymbol{u}_h}{\partial t}, \boldsymbol{v}_h\right) + \nu_{\mathrm{S}} a_h^{\mathrm{S}}(\boldsymbol{u}_h, \boldsymbol{v}_h) + c_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) - d(\boldsymbol{B}_h; \boldsymbol{B}_h, \boldsymbol{v}_h) \\
\qquad\qquad + J_h(\boldsymbol{B}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \ \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_k^h, \\
b(\boldsymbol{u}_h, q_h) = 0 \ \ \forall q_h \in Q_k^h, \\
\left(\dfrac{\partial \boldsymbol{B}_h}{\partial t}, \boldsymbol{H}_h\right) + \nu_{\mathrm{M}} a^{\mathrm{M}}(\boldsymbol{B}_h, \boldsymbol{H}_h) + d(\boldsymbol{B}_h; \boldsymbol{H}_h, \boldsymbol{u}_h) + \\
\qquad\qquad + K_h(\boldsymbol{u}_h; \boldsymbol{B}_h, \boldsymbol{H}_h) - b(\boldsymbol{H}_h, \varphi_h) = (\boldsymbol{G}, \boldsymbol{H}_h) \ \ \forall \boldsymbol{H}_h \in \boldsymbol{W}_k^h, \\
Y_h(\varphi_h, \psi_h) + b(\boldsymbol{B}_h, \psi_h) = 0 \ \ \forall \psi_h \in R_k^h,
\end{cases}
$$

$$
K_h(\chi; \boldsymbol{B}_h, \boldsymbol{H}_h) = \mu_K \sum_{f \in \Sigma_h^{\mathrm{int}}} h_f^2 \max\{\|\chi\|^2_{L^\infty(\omega_f)}, 1\} ([\![\nabla \boldsymbol{B}_h]\!]_f, [\![\nabla \boldsymbol{H}_h]\!]_f)_f \,.
$$

Theoretical results (nonlinear case, type 2)

Under standard mesh shape regularity, it holds

$$\|(\boldsymbol{u}-\boldsymbol{u}_h)(\cdot,T)\|^2+\|(\boldsymbol{B}_h-\boldsymbol{B}_h)(\cdot,T)\|^2+$$
$$+\int_0^T\|(\boldsymbol{u}-\boldsymbol{u}_h)(t)\|_{\mathrm{stab}}^2\,\mathrm{d}t+\int_0^T\|(\boldsymbol{B}-\boldsymbol{B}_h)(\cdot,t)\|_{4\mathrm{f}}^2\,\mathrm{d}t+\int_0^T|\varphi_h(\cdot,t)|_{Y_h}^2\,\mathrm{d}t\lesssim$$
$$(\Lambda_{\mathrm{stab}}^2\|\boldsymbol{u}\|^2_{L^2(0,T;\boldsymbol{W}_\infty^{k+1}(\Omega_h))}+\Lambda_{4\mathrm{f}}^2\|\boldsymbol{B}\|^2_{L^2(0,T;\boldsymbol{H}^{k+1}(\Omega_h))})h^{2k}+$$
$$+(\|\boldsymbol{u}\|^2_{H^1(0,T;\boldsymbol{H}^{k+1}(\Omega_h))}+\|\boldsymbol{B}\|^2_{H^1(0,T;\boldsymbol{H}^{k+1}(\Omega_h))})h^{2k+2}\,,$$

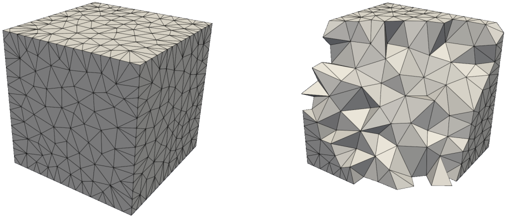
where

$$\Lambda_{4\mathrm{f}}^2:=\max\{\nu_{\mathrm{M}}\,,h(\gamma_{\mathrm{data}}^2+1)\}\,.$$

- Quasi-robust and pressure-robust;
- enjoys the $O(h^{k+1/2})$ pre-asymptotic error reduction in convective regimes.

A “basic” numerical test

We consider a model problem on a unitary cube, time interaval [0, 1], with known regular solution.



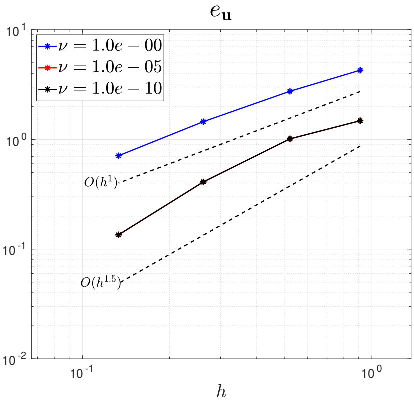
Considered space-time (squared) norms:

$$e_{\boldsymbol{u}}^2 = \|\boldsymbol{u}(\cdot, T) - \boldsymbol{u}_h(\cdot, T)\|_0^2 + \int_0^T \|\boldsymbol{u}(\cdot, t) - \boldsymbol{u}_h(\cdot, t)\|_{\text{stab}}^2 \, \mathrm{d}t, ,$$

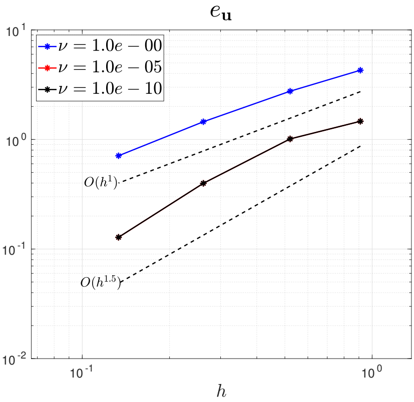
$$e_p^2 = \left(\int_0^T \|p(\cdot, t) - p_h(\cdot, t)\|_0^2 \, \mathrm{d}t \right) ,$$

$$e_{\boldsymbol{B}}^2 = \|\boldsymbol{B}(\cdot, T) - \boldsymbol{B}_h(\cdot, T)\|_0^2 + \int_0^T \|\boldsymbol{B}(\cdot, t) - \boldsymbol{B}_h(\cdot, t)\|_M^2 \, \mathrm{d}t \, .$$

Velocity field error

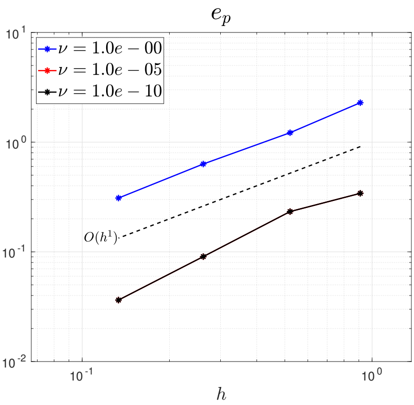


Type 1 Method

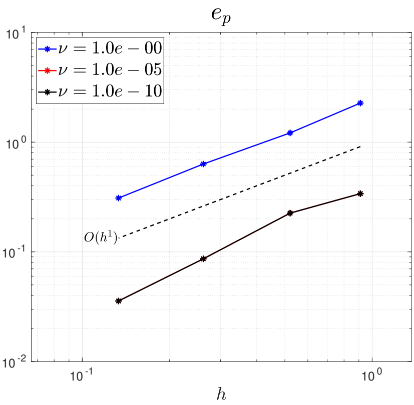


Type 2 Method

Pressure field error

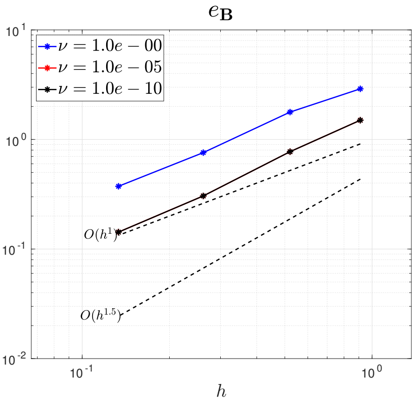


Type 1 Method

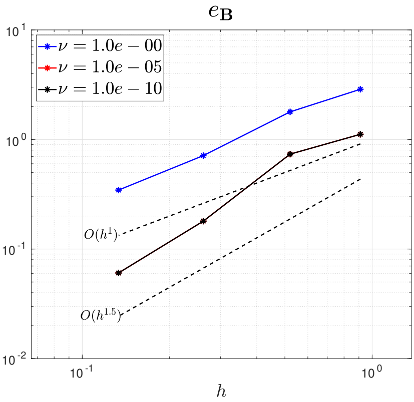


Type 2 Method

Magnetic field error



Type 1 Method



Type 2 Method

Variational formulation of the four field equations

Find $\mathbf{u} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\mathbf{H}^1_0(\Omega))$, $p \in L^2(0,T;L^2_0(\Omega))$,
$\mathbf{E} \in L^2(0,T;H_0(\mathbf{curl},\Omega))$ $\mathbf{B} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0(div,\Omega))$
such that for a.e. $t \in [0,T]$

$$\begin{cases} (\mathbf{u}_t,\mathbf{v}) + \nu_S a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) + c(\mathbf{u};\mathbf{u},\mathbf{v}) - (\mathbf{j}\times\mathbf{B},\mathbf{v}) = (\mathbf{f},\mathbf{v}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall \mathbf{v} \in \left[H^1_0(\Omega)\right]^3 \\ (\mathbf{j},\mathbf{F}) - \nu_M(\mathbf{B},\mathbf{curl}\mathbf{F}) = 0 \quad \forall \mathbf{F} \in H_0(\mathbf{curl},\Omega) \\ (\mathbf{B}_t,\mathbf{C}) + (\mathbf{curl}\mathbf{E},\mathbf{C}) = 0 \quad \forall \mathbf{C} \in H_0(div,\Omega) \\ b(\mathbf{u},q) = 0 \quad \forall q \in L_0(\Omega)\,, \end{cases}$$

where $\mathbf{j} = \mathbf{E} + \mathbf{u}\times\mathbf{B}$ and

$$a(\mathbf{u},\mathbf{v}) = (\nabla\mathbf{u},\nabla\mathbf{v})\,, \quad b(\mathbf{v},q) = -(div\mathbf{v},q)$$
$$c(\mathbf{w};\mathbf{u},\mathbf{v}) = ((\nabla\mathbf{u})\mathbf{w},\mathbf{v})\,.$$

Note: plus initial conditions, satisfying $div\mathbf{B}(0,\cdot) = 0$.

Two Exact Complexes

$$0 \xrightarrow{i} H_0^1(\Omega) \xrightarrow{\nabla} H_0(\mathbf{curl}, \Omega) \xrightarrow{\mathbf{curl}} H_0(div, \Omega) \xrightarrow{div} L_0^2(\Omega) \xrightarrow{o} 0$$

$$0 \xrightarrow{i} H_0^2(\Omega) \xrightarrow{\nabla} \mathbf{H}_0^1(\Omega) \xrightarrow{div} H^1(\Omega)/\mathbb{R} \xrightarrow{o} 0$$

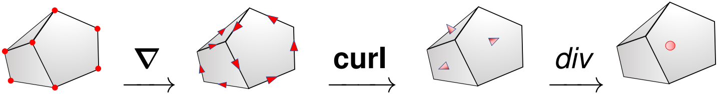
Two Exact Complexes

$$0 \xrightarrow{\ i\ } H^1_0(\Omega) \xrightarrow{\ \nabla\ } H_0(\mathbf{curl},\Omega) \xrightarrow{\ \mathbf{curl}\ } H_0(div,\Omega) \xrightarrow{\ div\ } L^2_0(\Omega) \xrightarrow{\ o\ } 0$$

$$0 \xrightarrow{\ i\ } H^2_0(\Omega) \xrightarrow{\ \nabla\ } \mathbf{H}^1_0(\Omega) \xrightarrow{\ div\ } H^1(\Omega)/\mathbb{R} \xrightarrow{\ o\ } 0$$

At the discrete level: **(talks by Mascotto & Dassi)**

$$\mathbf{V}_h^{NODE} \xrightarrow{\ \nabla\ } \mathbf{V}_h^{edge} \xrightarrow{\ \mathbf{curl}\ } \mathbf{V}_h^{face} \xrightarrow{\ div\ } \mathbf{V}_h^{ELEM}$$



$$0 \xrightarrow{\ i\ } S_h \xrightarrow{\ \nabla\ } \mathbf{W}_h \xrightarrow{\ div\ } Q_h \xrightarrow{\ o\ } 0$$

[LBdV, Brezzi, Dassi, Marini, Russo, SINUM & CMAME, 2018]
[LBdV, Lovadina, Vacca, M2AN 2017] [LBdV, Dassi, Vacca, M3AS 2020]

A VEM formulation

Find $(\mathbf{u}_h, p_h, \mathbf{E}_h, \mathbf{B}_h)$ in $\mathbf{W}_h \times Q_h \times \mathbf{V}_h^{edge} \times \mathbf{V}_h^{face}$ such that for a.e. $t \in I$

$$\begin{cases} m_h(\mathbf{u}_{h,t}, \mathbf{v}_h) + \nu_S\, a_h(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) + \widetilde{c}_h(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) \\ \qquad\qquad + \left[\mathbf{j}_h, \chi(\mathbf{v}_h, \mathbf{B}_h)\right]_{edge} = (\mathbf{f}, \Pi_1^0 \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{W}_h, \\ \left[\mathbf{j}_h, \mathbf{F}_h\right]_{edge} - \nu_M \left[\mathbf{B}_h, \mathbf{curl} \mathbf{F}_h\right]_{face} = 0 \qquad \forall \mathbf{F}_h \in \mathbf{V}_h^{edge}, \\ \left[\mathbf{B}_{h,t}, \mathbf{C}_h\right]_{face} + \left[\mathbf{curl} \mathbf{E}_h, \mathbf{C}_h\right]_{face} = 0 \qquad \forall \mathbf{C}_h \in \mathbf{V}_h^{face}, \\ b(\mathbf{u}_h, q_h) = 0 \qquad \forall q_h \in Q_h. \end{cases}$$

- Formulation for FEMs in [Hu, Ma, Xu, Numer. Math. 2017] (without an Ex. Complex for fluid part)
- VEM scalar products and discrete forms appearing above
- Preserves *div* $\mathbf{u} = 0$ and *div* $\mathbf{B} = 0$ exactly.
- Article is for lowest order VE spaces (extendable ...)

A convergence result

Let [BdV, Dassi, Manzini, Mascotto, M3AS 2023]

$$\mathbf{u} \in L^2(0,T;[H^2(\Omega)]^3)\,, \quad \partial_t \mathbf{u}, \mathbf{E}, \mathbf{f} \in L^2(0,T;[H^1(\Omega)]^3)\,,$$
$$\mathbf{B}, \mathbf{j} \in L^2(0,T;[H^1(\Omega) \cap L^\infty(\Omega)]^3)\,.$$

Then it holds

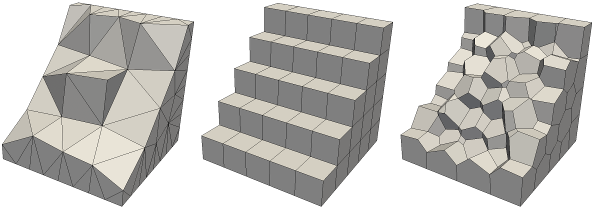
$$\|\mathbf{u}(t) - \mathbf{u}_h(t)\| + \|\mathbf{B}(t) - \mathbf{B}_h(t)\| + \left(\int_0^t \|\mathbf{E} - \mathbf{E}_h\|^2\right)^{\frac{1}{2}} \leq C\,h\,.$$

with C indep. of h and a.e. $t \in [0,T]$.

- Holds under standard mesh assumptions
- The analysis applies also to the FEM case (no convergence theory was developed in the FEM case for this approach)
- Estimates do not depend on pressure
- Estimates are not ν-robust

Numerical tests

Mesh families:



	$\|divB_h\|$			
	level 1	level 2	level 3	level 4
tetra	4.6977e-13	7.8962e-13	3.3856e-12	1.6680e-11
cube	1.1798e-13	2.2284e-13	6.6499e-13	2.2580e-12
voro	2.9645e-11	6.4690e-13	1.7873e-11	5.3376e-11

	$\|div\mathbf{u}_h\|$			
	level 1	level 2	level 3	level 4
tetra	7.6676e-16	2.0355e-15	1.0726e-14	6.1851e-14
cube	1.1714e-15	1.9226e-15	7.3605e-15	4.0470e-14
voro	2.7855e-16	3.2315e-15	1.6780e-14	8.8101e-14

- Standard error plots comply to the theory (see article).

Conclusions (RC4)

- We presented Research Cluster 4, focused on Geophysical Flows and Magnetohydrodynamics
- We have briefly presented the area of Magnetohydrodynamics in Numerical Analysis
- We have shown two stabilized Finite Element Methods that are pressure robust and convection quasi-robust (3-field and 4-field) for the fully nonlinear non-stationary model
- The 4-field method enjoyed also an improved pre-asymptotic error reduction rate in convection dominated regimes
- We have furthermore presented a VEM approach, based on Virtual Element Complexes, for a different four field formulation of the same model; both solenoidal constraints are satisfied "exactly".

Appendix: error norms

Velocity field:

$$\|\boldsymbol{u}\|_{1,h}^2 := \|\varepsilon_h(\boldsymbol{u})\|^2 + \mu_a \sum_{f\in\Sigma_h} h_f^{-1} \|[\![\, \boldsymbol{u} \,]\!]_f\|_f^2$$

$$|\boldsymbol{u}|_{\mathrm{upw},\boldsymbol{u}_\mathrm{h}}^2 := \sum_{f\in\Sigma_h^{\mathrm{int}}} \||\boldsymbol{u}_h \cdot \boldsymbol{n}_f|^{1/2} [\![\, \boldsymbol{u} \,]\!]_f\|_f^2$$

$$|\boldsymbol{u}|_{J_h,\boldsymbol{B}_h}^2 := \sum_{f\in\Sigma_h^{\mathrm{int}}} \max\{\|\boldsymbol{B}_h\|_{\boldsymbol{L}^\infty(\omega_f)}^2, 1\} \left(\|[\![\, \boldsymbol{u} \,]\!]_f\|_f^2 + h_f^2 \|[\![\, \nabla_h \boldsymbol{u} \,]\!]_f\|_f^2 \right)$$

$$\|\boldsymbol{u}\|_{\mathrm{stab}}^2 := \nu_\mathrm{S} \|\boldsymbol{u}\|_{1,h}^2 + |\boldsymbol{u}|_{\mathrm{upw},\boldsymbol{u}_\mathrm{h}}^2 + |\boldsymbol{u}|_{J_h,\boldsymbol{B}_h}^2 \, .$$

Magnetic field (type 1):

$$\|\boldsymbol{w}\|_M := \nu_M \|\nabla \boldsymbol{w}\|^2 + \|\mathrm{div}\boldsymbol{w}\|^2 \, .$$

Magnetic field (type 2):

$$\|\boldsymbol{w}\|_M := \nu_M \|\nabla \boldsymbol{w}\|^2 + \mu_K \sum_{f\in\Sigma_h^{\mathrm{int}}} \max\left\{1, \|\boldsymbol{u}_h\|_{L^\infty(\omega_f)}^2\right\} h_f^2 ([\![\, \nabla \boldsymbol{w} \,]\!] [\![\, \nabla \boldsymbol{w} \,]\!]|)_f \, .$$